CHAPTER 7

Spectral Theory

This chapter is devoted to a branch of the theory of operators very important for applications — spectral theory. More than any other chapter of the present book spectral theory owes its creation and intensive development to problems in natural sciences, in particular, in mechanics, physics, and chemistry.

7.1. The Spectrum of an Operator

The main object of spectral theory is the spectrum of a linear operator. Let X be a Banach space. A bounded operator $A\colon\ X \to X$ is called *invertible* if it maps X one-to-one onto X. By the Banach theorem the inverse mapping A^{-1} is automatically continuous. As in linear algebra, an important role is played by the property of invertibility of the operator $A - \lambda I$ for various scalars λ, where $I\colon\ x \mapsto x$ is the unit operator (the identity mapping).

7.1.1. Definition. *The spectrum $\sigma(A)$ of a bounded linear operator A on a complex Banach space X consists of all $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not invertible.*

For an operator on a real space similarly one defines the real spectrum. If $X = \{0\}$, then the only operator is zero; it has also zero inverse, hence its spectrum is empty. Usually this case is excluded from consideration; in the sequel we also do not always explicitly state that a nonzero space is in question.

The complement of the spectrum is called the *resolvent set* of the operator A and denoted by $\varrho(A)$. The points of the resolvent set are called *regular* points. For every $\lambda \in \varrho(A)$ the operator

$$R_\lambda(A) := (A - \lambda I)^{-1}$$

is called the *resolvent* of A (one should bear in mind that sometimes the resolvent is defined as the inverse to $\lambda I - A$). For $\lambda, \mu \in \varrho(A)$ we have the *Hilbert identity*

$$R_\lambda(A) - R_\mu(A) = (\lambda - \mu) R_\mu(A) R_\lambda(A),$$

which is easily verified by multiplying both sides by $(A - \lambda I)$ from the right and then multiplying by $(A - \mu I)$ from the left.

By Banach's inverse mapping theorem a point λ belongs to the spectrum if and only if either $\mathrm{Ker}(A - \lambda I) \neq 0$ or $(A - \lambda I)(X) \neq X$, where

$$\mathrm{Ker}(A - \lambda I) := \{x\colon\ Ax - \lambda x = 0\}.$$

V. I. Bogachev, O. G. Smolyanov, *Real and Functional Analysis*,
Moscow Lectures 4, https://doi.org/10.1007/978-3-030-38219-3_7

In the first case λ is an *eigenvalue*, i.e., $Av = \lambda v$ for some vector $v \neq 0$ (called an *eigenvector*). In the finite-dimensional space both cases can happen only simultaneously, but in infinite-dimensional spaces the situation is different.

7.1.2. Example. The operator $Ax = (0, x_1, x_2, \ldots)$ on l^2 is injective, but not surjective. The operator $Bx = (x_2, x_3, \ldots)$ on l^2 is surjective, but not injective. In both cases 0 belongs to the spectrum, but for different reasons. In addition, A has no eigenvalues, but one can show that $\sigma(A) = \{\lambda \in \mathbb{C}\colon |\lambda| \leqslant 1\}$. The Volterra operator on $L^2[0,1]$ or $C[0,1]$ (Example 6.9.4(iv)) also has no eigenvalues (if $Vx = \lambda x$, then $x(t) = \lambda x'(t)$, $x(0) = 0$).

As we shall see, the spectrum of every bounded operator (on a nonzero complex space) is a nonempty compact set. First we establish the following important fact.

7.1.3. Theorem. *The set of invertible operators on a Banach space X (complex or real) is open in the space $\mathcal{L}(X)$ with the operator norm. Moreover, if an operator $A \in \mathcal{L}(X)$ is invertible and $D \in \mathcal{L}(X)$ is an operator such that $\|D\| < 1/\|A^{-1}\|$, then the operator $A + D$ is invertible.*

PROOF. By Banach's theorem it suffices to show that for every $y \in X$ the equation $Ax + Dx = y$ is uniquely solvable. This equation is equivalent to the equation

$$A^{-1}(A + D)x = A^{-1}y,$$

which can be written as $A^{-1}y - A^{-1}Dx = x$. Set $F(x) = A^{-1}y - A^{-1}Dx$ and observe that F is a contracting mapping, since

$$\|F(x) - F(z)\| = \|A^{-1}D(x - z)\| \leqslant \|A^{-1}\|\|D\|\|x - z\|,$$

where $\|A^{-1}\|\|D\| < 1$. An alternative proof: if $\|D\| < 1$, then $\sum_{k=0}^{\infty} D^k$ converges in the operator norm and gives $(I - D)^{-1}$. Now $A + D = (I + DA^{-1})A$, hence $(A + D)^{-1}$ is given by $A^{-1} \sum_{k=0}^{\infty} (-DA^{-1})^k$. □

It follows from the theorem that the resolvent set is open. This assertion can be specified as follows.

7.1.4. Corollary. (i) *Let $A \in \mathcal{L}(X)$. Then, whenever $|\lambda| > \|A\|$, we have $\lambda \in \varrho(A)$ and*

$$R_\lambda(A) = -\sum_{k=0}^{\infty} \frac{A^k}{\lambda^{1+k}},$$

where the series converges in the operator norm.

(ii) *For every point $\lambda_0 \in \varrho(A)$, whenever $|\lambda - \lambda_0| < \|R_{\lambda_0}(A)\|^{-1}$, we have $\lambda \in \varrho(A)$ and*

$$R_\lambda(A) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}(A)^{k+1},$$

where the series converges in the operator norm.

PROOF. (i) We have $A - \lambda I = -\lambda I + A$, where $\|A\| < |\lambda| = 1/\|(\lambda I)^{-1}\|$. Convergence of the series of $-\lambda^{-1-k}A^k$ in the operator norm is obvious from the estimate $\|\lambda^{-k}A^k\| \leqslant |\lambda|^{-k}\|A\|^k$. It is straightforward to show that for its sum S_λ we have $S_\lambda(A-\lambda I) = (A-\lambda I)S_\lambda = I$.

(ii) Convergence of the series with respect to the norm is justified similarly. For its sum S_λ we have

$$\begin{aligned} S_\lambda(A-\lambda I) &= \sum_{k=0}^{\infty}(\lambda-\lambda_0)^k R_{\lambda_0}(A)^{k+1}\bigl(A-\lambda_0 I-(\lambda-\lambda_0)I\bigr) \\ &= \sum_{k=0}^{\infty}\bigl[(\lambda-\lambda_0)^k R_{\lambda_0}(A)^k-(\lambda-\lambda_0)^{k+1}R_{\lambda_0}(A)^{k+1}\bigr] = I. \end{aligned}$$

Similarly, $(A-\lambda I)S_\lambda = I$. □

7.1.5. Remark. If $\dim X < \infty$, then the set of invertible operators is not only open but is dense in $\mathcal{L}(X)$. In case of l^2 this is not true: the shift operator $A\colon (x_n) \mapsto (0, x_1, x_2, \ldots)$ cannot be approximated by invertible operators. Indeed, if $\|A-B\| < 1$, then B cannot be invertible, since by the equality $A^*A = I$ we have $\|I - A^*B\| \leqslant \|A^*\|\,\|A-B\| < 1$, which by the theorem above gives the invertibility of A^*B. If B were invertible, then also A^* would be invertible, hence also A. In this relation see also Exercises 7.10.112 and 7.10.113.

7.1.6. Theorem. *The spectrum of every operator $A \in \mathcal{L}(X)$ on a complex Banach space $X \neq 0$ is a nonempty compact set in the disc of radius $\|A\|$ centered at the origin in the complex plane.*

PROOF. The inclusion $\sigma(A) \subset \{z \in \mathbb{C}\colon |z| \leqslant \|A\|\}$ and the closedness of $\sigma(A)$ are already known. Let us verify that $\sigma(A)$ is not empty. Suppose that $R_\lambda(A)$ exists for all $\lambda \in \mathbb{C}$. Let $\psi \in \mathcal{L}(X)^*$ and $F(\lambda) = \psi\bigl(R_\lambda(A)\bigr)$. By assertion (ii) of the previous corollary F is an entire function, and by assertion (i) whenever $|\lambda| \to \infty$ we have $|F(\lambda)| \to 0$. By the Liouville theorem $F \equiv 0$, whence we obtain $R_\lambda(A) = 0$, which is impossible if $X \neq 0$. □

The obtained estimate of the radius of a disc containing the spectrum can be sharpened.

The *spectral radius* of the operator A is defined by the formula

$$r(A) := \inf\bigl\{\|A^n\|^{1/n}\colon\ n \in \mathbb{N}\bigr\}.$$

It is clear that $r(A) \leqslant \|A\|$, since $\|A^n\| \leqslant \|A\|^n$.

7.1.7. Proposition. *We have*

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

In addition, $r(A) = \max\{|z|\colon\ z \in \sigma(A)\}$.

PROOF. Let $\varepsilon > 0$. Let $p \in \mathbb{N}$ be such that $\|A^p\|^{1/p} \leqslant r(A) + \varepsilon$. If $n \geqslant p$, we have $n = kp + m$, where $0 \leqslant m \leqslant p-1$. Then

$$\|A^n\| \leqslant \|A^p\|^k\|A^m\| \leqslant M\|A^p\|^k, \quad M := 1 + \|A\| + \cdots + \|A^{p-1}\|.$$

Therefore,

$$r(A) \leqslant \|A^n\|^{1/n} \leqslant M^{1/n}\|A^p\|^{k/n} \leqslant M^{1/n}\bigl(r(A)+\varepsilon\bigr)^{kp/n}.$$

Since $M^{1/n} \to 1$ and $kp/n \to 1$ as $n \to \infty$, we have
$$r(A) \leqslant \limsup_{n\to\infty} \|A^n\|^{1/n} \leqslant r(A) + \varepsilon.$$
Since ε was arbitrary, this proves the first assertion.

Let us show that, whenever $|\lambda| > r(A)$, the operator $A - \lambda I$ is invertible. Dividing by λ, we arrive at the case $\lambda = 1$ and $r(A) < 1$. In this case the series $\sum_{n=0}^{\infty} A^n$ converges in the operator norm, since for all sufficiently large n we have $\|A^n\| \leqslant \bigl(r(A) + \varepsilon\bigr)^n$, where $\varepsilon > 0$ is such that $r(A) + \varepsilon < 1$. A straightforward verification shows that the sum of the indicated series serves as the inverse operator to $I - A$. It remains to show that the disc of radius $r(A)$ contains at least one point of the spectrum. Otherwise by the compactness of the spectrum we could find $r < r(A)$ such that all λ with $|\lambda| > r$ would belong to the resolvent set. According to the corollary proved above this means that for every continuous linear functional ψ on $\mathcal{L}(X)$ the function $f(\lambda) := \psi\bigl(R_\lambda(A)\bigr)$ is holomorphic on the set $|\lambda| > r$. Outside the disc of radius $\|A\|$ this function is represented by the Laurent series $-\sum_{k=0}^{\infty} \lambda^{-1-k}\psi(A^k)$. By the uniqueness of expansion the same series represents the function f if $|\lambda| > r$. Let us fix such $\lambda \in \bigl(r, r(A)\bigr)$. Convergence of the indicated Laurent series for every ψ gives the estimate $\sup_k \|\lambda^{-1-k}A^k\| \leqslant C < \infty$ by the Banach–Steinhaus theorem. Thus, $\|A^k\|^{1/k} \leqslant C^{1/k}\lambda^{1+1/k}$, whence $r(A) \leqslant \lambda$, which is a contradiction. $\square$

In the infinite-dimensional case very different operators can have equal spectra. Let us consider examples.

7.1.8. Example. Let $\{r_n\}$ be all rational numbers in $[0, 1]$, let A be the operator on l^2 given by the formula $Ax = (r_1x_1, r_2x_2, \ldots)$, and let B be the operator on $L^2[0, 1]$ given by the formula $Bx(t) = tx(t)$. Then both operators have the spectrum $[0, 1]$, although for A all numbers r_n are eigenvalues, while B has no eigenvalues. Indeed, $\{r_n\} \subset \sigma(A)$, whence $[0, 1] \subset \sigma(A)$ by the closedness of the spectrum. If $\lambda \notin [0, 1]$, then there exists the inverse operator $R_\lambda(A)x = \bigl((r_1 - \lambda)^{-1}x_1, (r_2 - \lambda)^{-1}x_2, \ldots\bigr)$.

Every point $\lambda \in [0, 1]$ belongs to $\sigma(B)$, since there is no function $x \in L^2[0, 1]$ such that $(t - \lambda)x(t) = 1$ a.e. If $\lambda \notin [0, 1]$, then the inverse operator for $B - \lambda I$ is the operator by multiplication by the bounded function $\varphi(t) = (t - \lambda)^{-1}$. The operator B has no eigenvalues: the equality $\lambda x(t) = tx(t)$ a.e. is only possible if $x(t) = 0$ a.e.

For an arbitrary linear operator A on a complex linear space and a polynomial $P(z) = \sum_{k=0}^{n} c_k z^k$ with complex coefficients, the operator $P(A)$ is defined by
$$P(A) = \sum_{k=0}^{n} c_k A^k, \quad A^0 := I.$$

7.1.9. Theorem. (THE SPECTRAL MAPPING THEOREM) *Let A be a bounded linear operator on a complex Banach space X. Then, for every polynomial P of complex variable, one has*
$$\sigma\bigl(P(A)\bigr) = P\bigl(\sigma(A)\bigr),$$
i.e., the spectrum of $P(A)$ is the image of the spectrum of A under the mapping P.

PROOF. Let us fix $\lambda \in \mathbb{C}$. Let $\lambda_1, \ldots, \lambda_n$ be the roots of the polynomial $P - \lambda$. Then $\lambda = P(\lambda_i)$ for all $i = 1, \ldots, n$, $P(z) - \lambda = c(z - \lambda_1) \cdots (z - \lambda_n)$ and

$$P(A) - \lambda I = c(A - \lambda_1 I) \cdots (A - \lambda_n I).$$

Let $c \neq 0$ (otherwise the assertion is obvious). We observe that the invertibility of $P(A) - \lambda I$ is equivalent to the invertibility of all operators $A - \lambda_i I$, since they commute. Indeed, if all these operators are invertible, then their product is also invertible. If some operator $A - \lambda_{i_0} I$ is not invertible, then either $\mathrm{Ker}(A - \lambda_{i_0} I) \neq 0$ or $(A - \lambda_{i_0} I)(X) \neq X$. Since by the commutativity of the factors we can put $A - \lambda_{i_0} I$ either on the first place or on the last one, the same relation is fulfilled for the whole product. Thus, λ belongs to $\sigma\big(P(A)\big)$ precisely when there exists a number i with $\lambda_i \in \sigma(A)$. The latter is equivalent to the property that $\lambda \in P\big(\sigma(A)\big)$. Indeed, if such i exists, then $\lambda = P(\lambda_i) \in P\big(\sigma(A)\big)$. If $\lambda = P(z)$, where $z \in \sigma(A)$, then z is one of the numbers λ_i, i.e., λ_i belongs to $\sigma(A)$. □

7.1.10. Remark. If $A \in \mathcal{L}(X)$, where X is a complex Banach space, then $\sigma(A) = \sigma(A^*)$ by Corollary 6.8.6(ii) and the equality $(A - \lambda I)^* = A^* - \lambda I$. However, for a Hilbert space X the spectrum $\sigma(A^*)$ is the set $\{\overline{z}\colon\ z \in \sigma(A)\}$ complex-conjugate to $\sigma(A)$, since here $(A - \lambda I)^* = A^* - \overline{\lambda} I$, because A^* acts on X. Note that in case of a Hilbert space the equivalence of the invertibility of the operators B and B^* is obvious from the equality $(BC)^* = C^* B^*$ and here Corollary 6.8.6 is not needed.

Let us consider the following important example.

7.1.11. Example. (THE OPERATOR OF MULTIPLICATION BY A FUNCTION) Let $\mu \neq 0$ be a finite nonnegative measure on a space Ω and let φ be a bounded complex μ-measurable function. Let us define the operator A_φ of multiplication by φ on $L^2(\mu)$ by the formula

$$A_\varphi x(\omega) = \varphi(\omega) x(\omega).$$

Then (i) A_φ^* is the operator of multiplication by the conjugate function $\overline{\varphi}$, the spectrum of A_φ is the *set of essential values* of φ, i.e., the set of all numbers $\lambda \in \mathbb{C}$ such that $\mu\big(\omega\colon\ |\varphi(\omega) - \lambda| \leqslant \varepsilon\big) > 0$ for all $\varepsilon > 0$;

(ii) $\varphi(\omega) \in \sigma(A_\varphi)$ for μ-a.e. ω;

(iii) $\|A_\varphi\| = \|\varphi\|_{L^\infty(\mu)}$.

In addition, the operator A_φ is selfadjoint precisely when $\varphi(\omega) \in \mathbb{R}^1$ for μ-a.e. ω.

PROOF. (i) The operator A_φ is bounded and $\|A_\varphi\| \leqslant \|\varphi\|_{L^\infty(\mu)}$. The expression for the adjoint to A_φ is clear from the equality

$$\int_\Omega \varphi(\omega) x(\omega) \overline{y(\omega)}\, \mu(d\omega) = \int_\Omega x(\omega) \overline{\overline{\varphi(\omega)} y(\omega)}\, \mu(d\omega)$$

for all $x, y \in L^2(\mu)$. Let us evaluate the spectrum of A_φ. If λ is not an essential value of φ, then for some $\varepsilon > 0$ for μ-almost all ω we have $|\varphi(\omega) - \lambda| \geqslant \varepsilon$. Redefining φ on a set of μ-measure zero, we can assume that this inequality is true for all $\omega \in \Omega$. Then the operator of multiplication by the bounded function $1/(\varphi - \lambda)$ is inverse to $A_\varphi - \lambda \cdot I$. Conversely, let λ be a regular value. If λ is an essential value of φ, then the sets $B_n = \{\omega\colon\ |\varphi(\omega)-\lambda| \leqslant 1/n\}$ have positive measures and hence the functions $x_n = I_{B_n}/\sqrt{\mu(B_n)}$ have unit norm. Moreover, $(A_\varphi - \lambda I)x_n \to 0$, since

$$\frac{1}{\mu(B_n)}\int_{B_n} |\varphi(\omega)-\lambda|^2\,\mu(d\omega) \leqslant n^{-2}\frac{1}{\mu(B_n)}\mu(B_n) = n^{-2}.$$

Then we obtain $x_n = (A_\varphi - \lambda I)^{-1}(A_\varphi - \lambda I)x_n \to 0$ contrary to the equality $\|x_n\| = 1$. (ii) The set S_φ of essential values of the function φ can differ from the set of its actual values. For example, let us define a function φ on the interval $(0,1)$ with Lebesgue measure as follows: $\varphi(t) = t$ if $t \neq 1/2$, $\varphi(1/2) = 2$. Then the value 2 is assumed, but is not essential, while the number 1 not belonging to the actual range of the function is essential. However, one can replace φ by a μ-almost everywhere equal function $\widetilde{\varphi}$ with values in the set of essential values of φ. Indeed, for every point $z \in \mathbb{C}$ that is not an essential value of φ we find an open disc $U(z,r)$ with $\mu\bigl(\varphi^{-1}\bigl(U(z,r)\bigr)\bigr) = 0$. The obtained cover of $\mathbb{C}^1\backslash S_\varphi$ contains a countable subcover by discs $U(z_j, r_j)$. The set $E = \bigcup_j \varphi^{-1}\bigl(U(z_j,r_j)\bigr)$ has μ-measure zero. In particular, there are essential values (otherwise $\mu(\Omega) = 0$). Outside E we make the function $\widetilde{\varphi}$ equal to φ and on E we make $\widetilde{\varphi}$ equal to some essential value. This modification takes values in S_φ. (iii) According to (i) it remains to show that $\|A_\varphi\| \geqslant \|\varphi\|_{L^\infty(\mu)}$. This is clear from the fact that the largest essential value of the function $|\varphi|$ is $\|\varphi\|_{L^\infty(\mu)}$. The equality $A_\varphi = A_\varphi^*$ is equivalent to the property that the operators of multiplication by φ and $\overline{\varphi}$ coincide, i.e., to the property that $\varphi(\omega) = \overline{\varphi(\omega)}$ for μ-a.e. ω. □

Note that the established fact does not extend to arbitrary infinite measures (Exercise 7.10.89). The importance of the considered example will be clear from the fact, which we prove later, that in such a form one can represent any selfadjoint operator and any unitary operator on a separable space.

If μ is Lebesgue measure on $[0,1]$ and $\varphi(\omega) = \omega$, then $\sigma(A_\varphi) = [0,1]$ and A_φ has no eigenvalues (Example 7.1.8). For a general Borel measure μ on $[a,b]$ and $\varphi(\omega) = \omega$, the spectrum of A_φ is the support of μ, i.e., $[a,b]$ without all intervals of μ-measure zero, and eigenvalues are points of positive μ-measure.

7.2. The Quadratic Form and Spectrum of a Selfadjoint Operator

For a continuous linear operator A on a complex Hilbert space H we define two functions

$$\Phi_A(x,y) = (Ax, y), \quad Q_A(x) = (Ax, x).$$

The function Q_A is called the *quadratic form* of the operator A. The identity

$$4\Phi_A(x,y) = Q_A(x+y) - Q_A(x-y) + iQ_A(x+iy) - iQ_A(x-iy)$$

yields that the function Φ_A and hence the operator A are uniquely determined by the quadratic form Q_A (in the real case this is false!). The function Φ_A, called the *bilinear form of the operator* A, is linear in the first argument, conjugate-linear in the second argument and continuous. Conversely, with the aid of such a function one can construct an operator generating this form.

7.2.1. Lemma. *Let H be a complex Hilbert space and let Φ be a complex function on $H \times H$ that is linear in the first argument, conjugate-linear in the second argument and continuous in every argument separately. Then there exists a continuous linear operator A on the space H such that $\Phi(u,v) = (Au,v)$ for all $u,v \in H$.*

PROOF. The mapping $v \mapsto \overline{\Phi(u,v)}$ is linear and continuous for any fixed vector u. The Riesz theorem gives a uniquely defined vector Au for which $(v,Au) = \overline{\Phi(u,v)}$, whence $(Au,v) = \Phi(u,v)$. The linearity of $\Phi(u,v)$ in u yields the linearity of the mapping $u \mapsto Au$. If $u_n \to 0$, then $\Phi(u_n,v) \to 0$ for all vectors $v \in H$, i.e., $Au_n \to 0$ weakly and hence $\{Au_n\}$ is bounded. Hence the operator A is continuous (see Theorem 6.1.3). $\square$

For a selfadjoint operator A the following important identity is valid:

$$4\mathrm{Re}(Ax,y) = Q_A(x+y) - Q_A(x-y).$$

For the proof it suffices to rewrite the expression

$$\bigl(A(x+y), x+y\bigr) - \bigl(A(x-y), x-y\bigr)$$

taking into account the equality $(Ay,x) = (y,Ax)$.

7.2.2. Lemma. *An operator A on a complex Hilbert space is selfadjoint precisely when its quadratic form Q_A is real.*

PROOF. If $A = A^*$, then

$$(Ax,x) = (x,Ax) = \overline{(Ax,x)}.$$

Conversely, if Q_A is a real function, then $Q_{A^*} = Q_A$, whence $A = A^*$, since the operator is uniquely determined by its quadratic form (recall that we consider complex spaces). $\square$

7.2.3. Theorem. (WEYL'S CRITERION) *A number λ belongs to the spectrum of a selfadjoint operator A precisely when there exists a sequence of vectors x_n such that*

$$\|x_n\| = 1 \quad \textit{and} \quad \|Ax_n - \lambda x_n\| \to 0.$$

PROOF. If such a sequence exists, then $\lambda \in \sigma(A)$, since otherwise

$$x_n = (A - \lambda I)^{-1}(A - \lambda I)x_n \to 0.$$

Suppose that there are no such sequences. Then

$$\inf_{\|x\|=1} \|Ax - \lambda x\| = \alpha > 0,$$

whence

$$\|Ax - \lambda x\| \geqslant \alpha \|x\| \quad \text{for all } x. \tag{7.2.1}$$

In particular, $\mathrm{Ker}(A - \lambda I) = 0$. Let us set $Y = (A - \lambda I)(H)$ and show that $Y = H$. Let $a \perp Y$, i.e., $(Ax - \lambda x, a) = 0$ for all x. Then $(x, Aa - \overline{\lambda}a) = 0$ and hence $Aa = \overline{\lambda}a$. If $a \neq 0$, then λ must be real, because Q_A is real. Hence $Aa = \lambda a$ contrary to the injectivity of $A - \lambda I$. Thus, the closure of Y coincides with H. Let $y \in H$. Pick $y_n = Ax_n - \lambda x_n \to y$. Using that $\{y_n\}$ is a Cauchy sequence and applying (7.2.1) we conclude that $\{x_n\}$ is a Cauchy sequence. By the completeness of H there exists $x = \lim_{n\to\infty} x_n$, whence $y = Ax - \lambda x$. Thus, the operator $A - \lambda I$ is invertible . $\square$

7.2.4. Corollary. *If A is a selfadjoint operator and λ is a complex number such that $\inf_{\|x\|=1} \|Ax - \lambda x\| > 0$, then λ is a regular number.*

We observe that Weyl's criterion and the previous corollary are also true in the real case (see also Exercise 7.10.97).

7.2.5. Corollary. *The spectrum of a selfadjoint operator is real.*

PROOF. If $\|x\| = 1$, for all real numbers α and β we have

$$\begin{aligned}&(Ax - \alpha x - i\beta x, Ax - \alpha x - i\beta x)\\ &= (Ax - \alpha x, Ax - \alpha x) - (Ax - \alpha x, i\beta x) - i\beta(x, Ax - \alpha x) + i\beta(x, i\beta x)\\ &\qquad = \|Ax - \alpha x\|^2 + \beta^2\|x\|^2 \geqslant \beta^2.\end{aligned}$$

If $\beta \neq 0$, then we apply the previous corollary. $\square$

7.2.6. Theorem. *For every selfadjoint operator A (on a nonzero complex or real Hilbert space) one has*

$$\|A\| = \sup\bigl\{|(Ax, x)|\colon\ \|x\| \leqslant 1\bigr\} = \sup\bigl\{|\lambda|\colon\ \lambda \textit{ is a point of the spectrum } A\bigr\}.$$

In addition, the spectrum of A contains the points

$$m_A = \inf_{\|x\|=1} (Ax, x), \quad M_A = \sup_{\|x\|=1} (Ax, x).$$

PROOF. Set $M = \sup_{\|x\|\leqslant 1} |(Ax, x)|$. It is clear that $M = M_A$ or $M = -m_A$. We have $|(Ax, x)| \leqslant M\|x\|^2$ and $M \leqslant \|A\|$, since $|(Ax, x)| \leqslant \|A\|\|x\|^2$. On the other hand,

$$\begin{aligned}\|A\| &= \sup_{\|x\|\leqslant 1} \|Ax\| = \sup_{\|x\|,\|y\|\leqslant 1} \operatorname{Re}(Ax, y)\\ &= \frac{1}{4} \sup_{\|x\|,\|y\|\leqslant 1} \bigl[(A(x + y), x + y) - (A(x - y), x - y)\bigr]\\ &\leqslant \frac{1}{4} \sup_{\|x\|,\|y\|\leqslant 1} \bigl[M\|x + y\|^2 + M\|x - y\|^2\bigr]\\ &= \frac{1}{2} \sup_{\|x\|,\|y\|\leqslant 1} \bigl[M\|x\|^2 + M\|y\|^2\bigr] = M.\end{aligned}$$

Thus, $M = \|A\|$. We can assume that $M = M_A$, because in the case $M = -m_A$ one can pass to the operator $-A$. Then there exist vectors x_n such that $\|x_n\| = 1$ and $(Ax_n, x_n) \to M$. This gives

$$\begin{aligned}\|Ax_n - Mx_n\|^2 &= (Ax_n, Ax_n) - 2M(Ax_n, x_n) + M^2(x_n, x_n)\\ &\leqslant \|A\|^2 + M^2 - 2M(Ax_n, x_n) = 2M^2 - 2M(Ax_n, x_n) \to 0.\end{aligned}$$

By Weyl's criterion we obtain $M \in \sigma(A)$. Taking into account that the spectrum is contained in the disc of radius $\|A\|$, this completes the proof of the equality indicated in the formulation and the inclusion $M_A \in \sigma(A)$ in case $M_A = M$.

For the proof of the inclusion $m_A, M_A \in \sigma(A)$ we observe that

$$M_{A+cI} = M_A + c,\ m_{A+cI} = m_A + c,\ \sigma(A + cI) = \sigma(A) + c$$

for every $c \in \mathbb{R}^1$. Let us take $c = \|A\|$. Since

$$|(Ax, x)| \leqslant \|A\|\|x\|^2,$$

we have $0 \leqslant m_{A+cI} \leqslant M_{A+cI}$, whence $M_{A+cI} \in \sigma(A+cI)$, i.e., $M_A \in \sigma(A)$ in any case. Finally, taking $c = -\|A\|$, on account of the equality $\sigma(-A) = -\sigma(A)$ we similarly obtain $m_A \in \sigma(A)$. □

7.2.7. Remark. It is obvious from our reasoning that

$$\sigma(A) \subset [m_A, M_A].$$

Of course, this follows at once from Weyl's criterion, since if $\|Ax_n - \lambda x_n\| \to 0$ and $\|x_n\| = 1$, then $(Ax_n, x_n) \to \lambda$.

For a selfadjoint operator A on a real Hilbert space H its complexification $A_{\mathbb{C}}$ on the complexification $H_{\mathbb{C}}$ of the space H acts by the natural formula $A_{\mathbb{C}}(x, iy) = (Ax, iAy)$ and, as one can easily see, is also a selfadjoint operator. The realification of this operator (passage to the field $\mathbb{R}$ forgetting the complex structure) is the direct sum of two copies of the operator A. It is readily verified that the spectra of A and $A_{\mathbb{C}}$ coincide.

If A, B are selfadjoint operators with $(Ax, x) \leqslant (Bx, x)$, then we write $A \leqslant B$ and $B \geqslant A$. In particular, $A \geqslant 0$ if $(Ax, x) \geqslant 0$ (as we know, in the complex case this estimate gives the selfadjointness of A, but in the real case the selfadjointness is required additionally). Such an operator is called *nonnegative* or *nonnegative definite*. It follows from what we have proved above that $A \geqslant 0$ precisely when $A = A^*$ and $\sigma(A) \subset [0, +\infty)$.

7.3. The Spectrum of a Compact Operator

Spectra of compact operators possess peculiar properties. Let X be a complex or real Banach space. Let us consider the operator $I - K$, where K is a compact operator.

7.3.1. Lemma. *Let K be a compact operator on X.*
(i) *The kernel of the operator $I - K$ is finite-dimensional.*
(ii) *The range of the operator $I - K$ is closed.*

PROOF. (i) On the kernel of the operator $I - K$ the operator I equals K and hence is compact, which is only possible if this kernel is finite-dimensional.

(ii) Let $y_n = x_n - Kx_n \to y$. We show that $y \in (I - K)(X)$. Suppose first that $\sup_n \|x_n\| < \infty$. By the compactness of K we can extract from $\{Kx_n\}$ a convergent subsequence $\{Kx_{n_i}\}$. Since $x_{n_i} = y_{n_i} + Kx_{n_i}$, the sequence $\{x_{n_i}\}$ converges as well. Denoting its limit by x, we obtain $y = x - Kx$.

We now consider the case where the sequence $\{x_n\}$ is not bounded. Set $Z = \mathrm{Ker}(I - K)$ and

$$d_n = \inf\{\|x_n - z\|\colon\ z \in Z\}.$$

Since Z is finite-dimensional, there exist vectors $z_n \in Z$ with $\|x_n - z_n\| = d_n$. We show that the sequence $\{d_n\}$ is bounded. Suppose the contrary. We can assume that $d_n \to +\infty$. Set

$$v_n = (x_n - z_n)/\|x_n - z_n\|.$$

Since $(I - K)z_n = 0$ and $\sup_n \|y_n\| < \infty$, we have

$$\|v_n\| = 1,\ v_n - Kv_n = (I - K)x_n/\|x_n - z_n\| = y_n/d_n \to 0.$$

The sequence $\{Kv_n\}$ contains a convergent subsequence $\{Kv_{n_i}\}$. Then $\{v_{n_i}\}$ converges to some vector $v \in X$. Moreover,

$$v - Kv = \lim_{i\to\infty} (v_{n_i} - Kv_{n_i}) = 0,$$

i.e., $v \in Z$. However, this is impossible, since $\mathrm{dist}(v, Z) \geqslant 1$, because

$$\|v_n - z\| = \frac{1}{d_n}\|x_n - z_n - d_n z\| \geqslant \frac{d_n}{d_n} = 1 \quad \text{for all } z \in Z,\ n \in \mathbb{N}.$$

Thus, the sequence $\{d_n\}$ is bounded. Now everything reduces to the first case, since $(I - K)(x_n - z_n) = (I - K)x_n = y_n$. □

Clearly, the lemma is also true for the operator $I + K$, since the operator $-K$ is compact too.

The next theorem is the main result of this section.

7.3.2. Theorem. *Let K be a compact operator on a complex or real infinite-dimensional Banach space X. Then the spectrum of K either coincides with the point 0 or has the form*

$$\sigma(K) = \{0\} \cup \{k_n\},$$

where all numbers k_n are eigenvalues of K of finite multiplicity, which means that $\dim \mathrm{Ker}\,(K - k_n I) < \infty$, and the collection $\{k_n\}$ is either finite or is a sequence converging to zero.

PROOF. By the noncompactness of I the operator K is not invertible and hence $0 \in \sigma(K)$. Let $\lambda \in \sigma(K)$ and $\lambda \neq 0$. We show that λ is an eigenvalue. Suppose the contrary. Passing to the operator $\lambda^{-1}K$, we can assume that $\lambda = 1$. By the lemma the subspace $X_1 = (K - I)(X)$ is closed in X. In addition, we have $X_1 \neq X$, since otherwise $K - I$ would be invertible. Set

$$X_n = (K - I)^n(X) = (K - I)(X_{n-1}), \quad n \geqslant 2.$$

It is clear that $X_{n+1} \subset X_n$, since $X_1 \subset X$, whence $X_2 \subset X_1$ and so on. By the lemma we obtain that all subspaces X_n are closed. They are all different by the injectivity of $K - I$, since if

$$(K - I)(X_n) = (K - I)(X_{n-1}),$$

then $X_n = X_{n-1}$, whence we obtain $X_n = \cdots = X_1 = X$.

According to Theorem 5.3.4 there exist vectors $x_n \in X_n$ such that $\|x_n\| = 1$ and $\operatorname{dist}(x_n, X_{n+1}) \geqslant 1/2$. If $n < m$, we have

$$Kx_n - Kx_m = x_n - x_m + (K - I)x_n - (K - I)x_m,$$

where

$$-x_m + (K - I)x_n - (K - I)x_m \in X_m + X_{n+1} + X_{m+1} \subset X_{n+1}.$$

Hence $\|Kx_n - Kx_m\| \geqslant 1/2$, i.e., $\{Kx_n\}$ contains no Cauchy subsequence contrary to the compactness of K. The obtained contradiction means that λ is an eigenvalue of K. By the lemma $\dim \operatorname{Ker}(K - \lambda I) < \infty$, i.e., λ has a finite multiplicity.

We now show that $\sigma(K)$ has no nonzero limit points. Suppose that $\lambda_n \to \lambda$, where λ_n are eigenvalues and $\lambda \neq 0$. We can assume that λ_n are distinct and $|\lambda_n| \geqslant \sigma > 0$. Let us take $x_n \neq 0$ with $Kx_n = \lambda_n x_n$. It is readily seen that the vectors x_n are linearly independent. Denote by X_n the linear span of $x_1, \ldots, x_n$. It is clear that $K(X_n) \subset X_n$. By Theorem 5.3.4 there exist $y_n \in X_n$ with $\|y_n\| = 1$ and $\operatorname{dist}(y_n, X_{n-1}) \geqslant 1/2$, $n > 1$. We have

$$y_n = \alpha_n x_n + z_n, \quad z_n \in X_{n-1}.$$

Then for $n > m$ we have

$$\begin{aligned} Ky_n - Ky_m &= K(\alpha_n x_n) + Kz_n - Ky_m = \alpha_n \lambda_n x_n + Kz_n - Ky_m \\ &= \lambda_n (y_n - z_n + \lambda_n^{-1} Kz_n - \lambda_n^{-1} Ky_m), \end{aligned}$$

where $-z_n + \lambda_n^{-1} Kz_n - \lambda_n^{-1} Ky_m \in X_{n-1}$, because $z_n \in X_{n-1}$, $Kz_n \in X_{n-1}$, $Ky_m \in X_m \subset X_{n-1}$. Since $|\lambda_n| \geqslant \sigma$ and $\operatorname{dist}(y_n, X_{n-1}) \geqslant 1/2$, we have $\|Ky_n - Ky_m\| \geqslant \sigma/2$. Hence $\{Ky_n\}$ contains no Cauchy subsequence, which is a contradiction. $\square$

7.3.3. Example. *The Volterra operator V on $L^2[0,1]$ or on $C[0,1]$ (see Example* 6.9.4(iv)) *has no eigenvalues, i.e., $\sigma(V) = \{0\}$.*

7.3.4. Corollary. *Let K be a compact operator on X. Then $(I - K)(X)$ is a closed subspace of finite codimension, i.e., $X = (I - K)(X) \oplus E$, where E is a finite-dimensional linear subspace.*

PROOF. The closedness of $(I - K)(X)$ is already established. According to Lemma 6.8.1 this subspace is the intersection of the kernels of the functionals in the kernel of $I - K^*$. Since $\dim \operatorname{Ker}(I - K^*) < \infty$ by the compactness of K^* (see Theorem 6.9.3), there are linearly independent functionals $l_1, \ldots, l_n \in \operatorname{Ker}(I - K^*)$ such that $(I - K)(X) = \bigcap_{i=1}^n \operatorname{Ker} l_i$. Let us take vectors $x_i \in X$ with $l_i(x_j) = \delta_{ij}$. Then X is the sum of $(I - K)(X)$ and the linear span of $x_1, \ldots, x_n$. Indeed, for every $x \in X$ we set $z = x - \sum_{i=1}^n l_i(x) x_i$. This gives $l_j(z) = l_j(x) - l_j(x) l_j(x_j) = 0$ for all $j = 1, \ldots, n$. $\square$

It is clear that this corollary remains in force for $\lambda I - K$ with $\lambda \neq 0$, since $\lambda^{-1} K$ is a compact operator. In the next section we use this observation.

7.4. The Fredholm Alternative

We already know that if a nonzero number λ is not an eigenvalue for a compact operator K on a Banach space X, then the operator $K - \lambda I$ is invertible and hence the equation

$$Kx - \lambda x = y \tag{7.4.1}$$

is uniquely solvable for every $y \in X$. Here we sharpen this assertion and show that the solvability of equation (7.4.1) for all y yields its unique solvability. In other words, the nontriviality of the kernel of $K - \lambda I$ means that $(K - \lambda I)(X) \neq X$, exactly as in the finite-dimensional case.

7.4.1. Theorem. (THE FREDHOLM ALTERNATIVE) *Let K be a compact operator on a complex or real Banach space X. Then*

$$\mathrm{Ker}(K - I) = 0 \iff (K - I)(X) = X,$$

i.e., either the equation

$$Kx - x = y$$

is uniquely solvable for all $y \in X$ or for some vector $y \in X$ it has no solutions and then the homogeneous equation

$$Kx - x = 0$$

has nonzero solutions.

PROOF. If $\mathrm{Ker}(K - I) = 0$, then by Theorem 7.3.2 we have $1 \notin \sigma(K)$. Hence $(K - I)(X) = X$. Conversely, suppose that

$$(K - I)(X) = X, \quad \text{but} \quad \mathrm{Ker}(K - I) \neq 0.$$

As we know, the operator K^* on X^* is also compact (Theorem 6.9.3). We observe that $\mathrm{Ker}\,(K^* - I) = 0$. Indeed, if $f \in X^*$ and $(K^* - I)f = 0$, then

$$f\big((K - I)x\big) = (K^* - I)f(x) = 0 \quad \text{for all } x \in X.$$

Since $(K - I)(X) = X$, we have $f = 0$. By Theorem 7.3.2 the operator $K^* - I$ is invertible. We now take a nonzero element $a \in \mathrm{Ker}(K - I)$. By the Hahn–Banach theorem there is a functional $f \in X^*$ with $f(a) = 1$. Let $g = (K^* - I)^{-1} f$. Then $(K^* - I)g(a) = f(a) = 1$. On the other hand, $(K^* - I)g(a) = g\big((K - I)a\big) = 0$, which is a contradiction. Part of this reasoning could be replaced by a reference to Corollary 6.8.6. $\square$

7.4.2. Corollary. *The Fredholm alternative remains in force also for an operator $K \in \mathcal{L}(X)$ such that for some $n \in \mathbb{N}$ the operator K^n is compact.*

PROOF. Let $1 \in \sigma(K)$. Since K^n is compact and $\sigma(K^n)$ is the image of $\sigma(K)$ under the mapping $z \mapsto z$, the unit circumference can contain only finitely many points $\lambda_1, \ldots, \lambda_m$ from $\sigma(K)$. Increasing n, we can assume that n is a simple number and $\exp(2k\pi i/n) \neq \lambda_j$ for $k = 1, \ldots, n-1$, $j = 1, \ldots, m$. Let

$\theta := \exp(2\pi i/n)$. Then θ^k differs from all λ_j with $k = 1, \dots, n-1$, i.e., the operators $I - \theta^k K$ are invertible. Hence the operator $V = (I - \theta K) \cdots (I - \theta^{n-1} K)$ is also invertible. Since

$$I - K^n = (I - K)(I - \theta K) \cdots (I - \theta^{n-1} K) = (I - K)V,$$

where V is invertible and commutes with K, we conclude that $K - I$ and $K^n - I$ have equal kernels and equal ranges. □

Clearly, the Fredholm alternative remains in force for K–λI for all $\lambda \neq 0$, i.e., the solvability of (7.4.1) with every right-hand side is equivalent to the absence of nontrivial solutions to the equation

$$Kx - \lambda x = 0. \tag{7.4.2}$$

As an application of the Fredholm alternative we prove the following important result due to Weyl on the behavior of spectra under compact perturbations.

7.4.3. Theorem. *Let X be a Banach space and let A be a bounded operator on X. Then for every compact operator K on X, the spectra of the operators A and $A + K$ coincide up to the sets of eigenvalues, i.e.,*

$$\sigma(A)\backslash\sigma_p(A) \subset \sigma(A + K) \quad \textit{and} \quad \sigma(A + K)\backslash\sigma_p(A + K) \subset \sigma(A),$$

where $\sigma_p(A)$ denotes the so-called point spectrum of A, i.e., the set of all eigenvalues.

PROOF. Let $\lambda \in \sigma(A)$. We have to show that if the operator $C := A+K-\lambda I$ is invertible, then λ is an eigenvalue of A. Let us consider the equality

$$A - \lambda I = C + (A - \lambda I - C) = C - K = C(I - C^{-1}K).$$

Since $\lambda \in \sigma(A)$, the operator $I - C^{-1}K$ cannot be invertible. By the Fredholm theorem (which can be applied due to the compactness of $C^{-1}K$), it has a nonzero kernel: there exists a nonzero vector v such that $C^{-1}Kv = v$. Then $Kv = Cv$, whence $Av = \lambda v$, as required. Applying this to the operators $A + K$ and $-K$, we obtain the second relation in the theorem. □

It is worth noting that the full spectra of A and $A + K$ can be still very different (Exercise 7.10.64).

The classic results of Fredholm were obtained in terms of integral equations. Before turning to their discussion, we include yet another abstract result, also belonging to the so-called Fredholm theorems and dealing with the connection between the solvability of equations of the form (7.4.1) and (7.4.2) and analogous equations with the adjoint operator. We recall that for any compact operator K on X the adjoint operator K^* on X^* is also compact. In addition, $\sigma(K^*) = \sigma(K)$, but if the space X is Hilbert and the adjoint operator K^* is considered on X, then we have $\sigma(K^*) = \overline{\sigma(K)}$.

We recall that by Lemma 6.8.1 the closure of the range $A(X)$ of a bounded operator is the intersection of kernels of functionals in the kernel of its adjoint $A(X)$. This gives the first assertion in the next theorem, since the range of $K - I$ is closed for a compact operator K by Lemma 7.3.1. But more can be said in this case.

7.4.4. Theorem. *Let $K \in \mathcal{K}(X)$, $\lambda \neq 0$. Equation* (7.4.1) *is solvable for those and only those y which belong to the set*

$$\bigl\{z \in X\colon\ f(z) = 0 \ \ \forall\, f \in \operatorname{Ker}(K^* - \lambda I)\bigr\} = \bigcap_{f \in \operatorname{Ker}(K^* - \lambda I)} \operatorname{Ker} f,$$

called the annihilator of $\operatorname{Ker}(K^* - \lambda I)$ *in X. In addition,*

$$\dim \operatorname{Ker}\,(K - \lambda I) = \dim \operatorname{Ker}\,(K^* - \lambda I) = \operatorname{codim}\,(K - \lambda I)(X). \qquad (7.4.3)$$

If X is Hilbert, then in place of $K^* - \lambda I$ we take $K^* - \overline{\lambda} I$.

PROOF. It suffices to consider $\lambda = 1$. The first assertion was explained in the proof of Corollary 7.3.4. It was shown there that there are vectors $x_1, \ldots, x_n$ in X and functionals $l_1, \ldots, l_n \in \operatorname{Ker}\,(K^* - I)$ such that the kernel $\operatorname{Ker}\,(K^* - I)$ coincides with the linear span of the functionals $l_1, \ldots, l_n$, $l_i(x_j) = \delta_{ij}$ for all i, j, $(K - I)(X) = \bigcap_{i=1}^n \operatorname{Ker} l_i$, and the n-dimensional subspace E generated by $x_1, \ldots, x_n$ complements the closed subspace $(K - I)(X)$ to X. Thus,

$$\dim \operatorname{Ker}\,(K^* - I) = \dim E = \operatorname{codim}\,(K - I)(X).$$

We now show that $\dim \operatorname{Ker}\,(K - I) = n$. The finite-dimensional subspace $X_0 := \operatorname{Ker}\,(K - I)$ can be complemented to X by a closed linear subspace X_1 (Corollary 6.4.2). If $\dim X_0 < n$, then we can find an injective, but not surjective operator $K_0\colon X_0 \to E$. Writing x in the form $x = x_0 \oplus x_1$, $x_0 \in X_0$, $x_1 \in X_1$, we obtain the operator $K_1\colon\ X \to X$, $x \mapsto K_0 x_0 + Kx$. This operator is compact by the compactness of K and the aforementioned corollary. The kernel of $K_1 - I$ is trivial: if $K_1 x = x$, then $x - Kx = K_0 x_0 \in E$, whence $K_0 x_0 = 0$ and $x_0 = 0$, since $E \cap (K - I)(X) = 0$ and $\operatorname{Ker} K_0 = 0$. This gives $x_1 - Kx_1 = 0$ and $x_1 = 0$ by the injectivity of $K - I$ on X_1. In addition, the range of $K_1 - I$ differs from X (not all vectors from E belong to it), which is impossible by the Fredholm alternative. Similarly, if $\dim X_0 > n$, then there is a surjective operator $K_0\colon\ X_0 \to E$ with a nonzero kernel. This gives a surjective operator $K_1 - I$ with a nonzero kernel, which is also impossible. Thus, $\dim X_0 = n$. By the already established facts the codimension of the range of $K^* - I$ is also n. □

Let us apply the established abstract results to the objects from which Fredholm's theory was beginning — integral operators. Suppose we are given a complex square integrable function (an integral kernel) $\mathcal{K}$ on $[a, b]^2$ or, more generally, on $\Omega \times \Omega$, where $(\Omega, \mathcal{A}, \mu)$ is a space with a nonnegative measure. This kernel defines a compact operator by the formula $Kx(t) = \displaystyle\int_a^b \mathcal{K}(t, s)x(s)\, ds$ on the complex space $L^2[a, b]$ or a similarly defined operator on $L^2(\mu)$. A straightforward calculation shows that the adjoint operator K^* is defined by the formula

$$K^*u(t) = \int_a^b \overline{\mathcal{K}(s, t)}u(s)\, ds,$$

i.e., corresponds to the integral kernel $\mathcal{K}^*(t, s) := \overline{\mathcal{K}(s, t)}$. If the kernel is real and symmetric, then $\mathcal{K} = \mathcal{K}^*$. The first question of the theory of integral equations concerns the solvability of the equation

$$x(t) - \int_a^b \mathcal{K}(t,s)x(s)\,ds = y(t) \tag{7.4.4}$$

with a given right-hand side. The results obtained above lead to the following conclusions concerning (7.4.4).

(1) The set of solutions to equation (7.4.4) with $y = 0$ has a finite dimension n, and the same dimension has the set of solutions to the homogeneous equation corresponding to the kernel $\mathcal{K}^*$;

(2) if $u_1, \ldots, u_n$ are linearly independent solutions to the homogeneous equation corresponding to $\mathcal{K}^*$, then the set of all those y for which equation (7.4.4) is solvable consists of the functions y for which $\displaystyle\int_a^b y(t)\overline{u_i(t)}\,dt = 0, \quad i = 1, \ldots, n.$

The same is also true for operators on real spaces given by real kernels.

Let us consider analogous questions for the integral operator K given by a continuous real kernel $\mathcal{K}$ on $C[a,b]$. This operator is also compact. Hence we can apply Theorem 7.4.4. However, this theorem employs the adjoint operator on the adjoint space $C[a,b]^*$, which is the space of measures. It is natural to ask whether in the study of the solvability problem for the equation $x - Kx = y$ it suffices to consider the adjoint operator

$$K'x(t) = \int_a^b \mathcal{K}(s,t)x(s)\,ds$$

only on functions from $C[a,b]$ (the operator K' is the restriction of K^* to the subspace in $C[a,b]^*$ corresponding to measures given by continuous densities). This turns out to be possible. Indeed, by the general theorem we have to investigate the equation $\sigma - K^*\sigma = 0$ in $C[a,b]^*$, where $K^*\sigma$ is the measure acting on functions $x \in C[a,b]$ by the formula

$$K^*\sigma(x) = \int_a^b Kx(t)\,\sigma(dt) = \int_a^b \int_a^b \mathcal{K}(t,s)x(s)\,ds\,\sigma(dt).$$

This means that the measure $K^*\sigma$ is given by the continuous density

$$\varrho(t) = \int_{[a,b]} \mathcal{K}(t,s)\,\sigma(ds).$$

Hence the existence of nontrivial solutions to the equation $\sigma - K^*\sigma = 0$ in $C[a,b]^*$ is equivalent to the existence of nonzero solutions to the equation $\varrho - K'\varrho = 0$ in $C[a,b]$. It should be noted that here we have used the continuity of $\mathcal{K}$ in both variables (more precisely, it is important here that K^* takes $C[a,b]^*$ to $C[a,b]$). Let us consider the kernel $\mathcal{K}(t,s) = (3/2)ts^{-1/2}$, which is continuous only with respect to one variable, but obviously generates a compact operator on $C[0,1]$ with a one-dimensional range. The operator K^* on $C[0,1]^*$ also has a one-dimensional range and the measure $\sigma = s^{-1/2}\,ds$ satisfies the equation $K^*\sigma = \sigma$ and spans the subspace Ker $(K^* - I)$. By Theorem 7.4.4 the condition for the solvability of the equation $x - Kx = y$ is given by the equality

$$\int_0^1 y(s)s^{-1/2}\,ds = 0.$$

If we act here by a formal analogy with the previous case and search eigenvectors of the adjoint kernel only in $C[0,1]$ or $L^2[0,1]$ (and not in $C[0,1]^*$), then, having seen that they are absent, we could arrive at the wrong conclusion about the solvability of the equation $x - Kx = y$ for all y.

Note that the continuity of the kernel $\mathcal{K}$ in the case of an interval or its square integrability in the general case were only needed to verify the compactness of K. Theorem 7.4.4 applies also to the operators on $L^2[0,1]$ or $C[0,1]$ given by singular kernels $\mathcal{K}(t,s) = \mathcal{K}_0(t,s)|t-s|^{-\alpha}$, where $\alpha < 1$ and a measurable function $\mathcal{K}_0$ is bounded in the case of $L^2[0,1]$ and bounded and continuous in t in the case of $C[0,1]$ (see Exercise 6.10.143). If the function $\mathcal{K}_0$ is continuous and $\alpha < 1/2$, then in the study of the solvability of the equation $x - Kx = y$ in $C[0,1]$ it is also sufficient to analyze the equation $z - K'z = 0$ corresponding to the adjoint kernel only in $C[0,1]$ and not in $C[0,1]^*$. Indeed, for any $y \in C[0,1]$ the solvability of the equation $x - Kx = y$ in $C[0,1]$ is equivalent to its solvability in $L^2[0,1]$, since $Kx \in C[0,1]$ for all $x \in L^2[0,1]$, which is easily verified by using the square integrability of $|s|^{-\alpha}$ with $\alpha < 1/2$. In addition, all solutions to the equation $z - K'z = 0$ in $L^2[0,1]$ also belong to $C[0,1]$.

7.5. The Hilbert–Schmidt Theorem

In a finite-dimensional space every selfadjoint operator is diagonal in some orthonormal al basis. In an infinite-dimensional case a selfadjoint operator can fail to have eigenvectors (for example, the operator $Ax(t) = tx(t)$ on $L^2[0,1]$). However, for compact selfadjoint operators there is a full analogy with the finite-dimensional case. This is asserted in the following remarkable classic result.

7.5.1. Theorem. (THE HILBERT–SCHMIDT THEOREM) *Suppose that A is a compact selfadjoint operator on a real or complex separable Hilbert space $H \neq 0$. Then A has an orthonormal eigenbasis $\{e_n\}$, i.e., $Ae_n = \alpha_n e_n$, where the numbers α_n are real and converge to zero if H is infinite-dimensional.*

PROOF. We observe that A has eigenvectors. Indeed, by Theorem 7.2.6 the infimum and supremum of the function $Q_A(x) = (Ax,x)$ on the unit sphere belong to the spectrum. If they are zero, then $A = 0$. If at least one of these numbers is not zero, by the compactness of A it is an eigenvalue. All eigenvalues of A are real. The eigenvectors corresponding to different eigenvalues are mutually orthogonal. Indeed, if $Aa = \alpha a$ and $Ab = \beta b$, then

$$\beta(a,b) = (a,Ab) = (Aa,b) = \alpha(a,b),$$

whence $(a,b) = 0$ if $\alpha \neq \beta$. Therefore, by the separability of H there are at most countably many eigenvalues. Every nonzero eigenvalue has a finite multiplicity by the compactness of A. Let $\{\alpha_n\}$ be all eigenvalues of A. In every subspace $H_n := \operatorname{Ker}(A - \alpha_n I)$ we can choose an orthonormal basis (for $\alpha_n \neq 0$ such bases are finite). The union of all these bases gives an orthonormal system $\{e_n\}$ in H. It remains to show that $\{e_n\}$ is a basis. Denote by H' the closed linear span of $\{e_n\}$. It is readily seen that $A(H') \subset H'$, since $A(H_n) \subset H_n$ for all n. Let H'' be the orthogonal complement of H'. We observe that $A(H'') \subset H''$.

Indeed, if $u \in H''$, then for every $v \in H'$ we have $(Au, v) = (u, Av) = 0$, since $Av \in H'$. As shown above, in H'' we have an eigenvector of A, which leads to a contradiction if $H'' \neq 0$. Theorem 7.3.2 gives $\alpha_n \to 0$. □

7.5.2. Remark. A similar assertion is true for nonseparable spaces H. Here there exists a separable closed subspace $H_0 \subset H$ such that $A(H_0) \subset H_0$ and $A(H_0^\perp) = 0$. For H_0 we can take the closure of $A(H)$, which is separable due to the compactness of A.

There is another (variational) proof of this important theorem, which does not use the spectral theory. The reasoning above shows that the main problem is to establish the existence of at least one eigenvector. Such a vector can be found by solving the maximization problem for the function $Q(x) = |(Ax, x)|$ on the closed unit ball (of course, if we already have the Hilbert–Schmidt theorem, then it is clear that the maximum is attained at the eigenvectors corresponding to the eigenvalues with the maximal absolute value). Denote by q the supremum of this function and take unit vectors h_n with $Q(h_n) \to q$. Pick a subsequence $\{h_{n_i}\}$ weakly converging to some vector h. The sequence $\{Ah_{n_i}\}$ converges in norm to Ah, whence $(Ah_{n_i}, h_{n_i}) \to (Ah, h)$ and $Q(h) = q$. It is clear that $\|h\| = 1$ if $\|A\| > 0$. It is now easy to verify that h is an eigenvector. For this it suffices to show that $Ah \perp h^\perp$, since in that case $Ah = \lambda h$. Let $e \perp h$, $\|e\| = 1$. We can assume that $(Ah, h) > 0$. Then for all real t we have $(1 + t^2)^{-1} Q(h + te) \leqslant q$, that is,

$$q + 2t\mathrm{Re}\,(Ah, e) + t^2 (Ae, e) \leqslant q + qt^2.$$

This is only possible if $\mathrm{Re}\,(Ah, e) = 0$. Since this is true for all $e \in h^\perp$, we obtain that $Ah \perp h^\perp$.

Note also that in place of $|(Ax, x)|$ we could search the maximum of the function (Ax, x), provided that we assume additionally that it assumes a positive value. If $(Ax, x) \leqslant 0$, then we can pass to $-A$.

The reasoning presented above not only gives another proof, but also leads to the following useful variational principle. Denote by $\mathcal{L}_n$ the collection of all n-dimensional linear subspaces in a Hilbert space H.

7.5.3. Theorem. *Let A be a compact selfadjoint operator on a real or complex Hilbert space and let $\alpha_1 \geqslant \alpha_2 \geqslant \cdots > 0$ be all positive eigenvalues of A written in the order of decreasing taking into account their multiplicities. Then the following equalities are valid for all n for which there exists $\alpha_n > 0$.*

(i) *The Courant variational principle*:

$$\alpha_n = \min_{L \in \mathcal{L}_{n-1}} \max_{x \in L^\perp,\, \|x\|=1} (Ax, x). \tag{7.5.1}$$

(ii) *The Fischer variational principle*:

$$\alpha_n = \max_{L \in \mathcal{L}_n} \min_{x \in L,\, \|x\|=1} (Ax, x). \tag{7.5.2}$$

PROOF. (i) Let $L \in \mathcal{L}_{n-1}$. In the space H_n generated by the orthonormal eigenvectors $e_1, \ldots, e_n$ corresponding to the eigenvalues $\alpha_1, \ldots, \alpha_n$, there is a unit vector $h \perp L$. Since for any $x \in H_n$ we have $(Ax, x) \geqslant \alpha_n(x, x)$, it follows that

$$\max_{x \in L^{\perp}, \|x\|=1} (Ax, x) \geqslant \alpha_n.$$

If we take $L = H_{n-1}$, then the equality is attained.

(ii) Let $L \in \mathcal{L}_n$. Then L contains a vector $h \perp H_{n-1}$, where H_n is the same as above. It is clear that $(Ah, h) \leqslant \alpha_n(h, h)$. Hence $\min_{x \in L, \|x\|=1}(Ax, x) \leqslant \alpha_n$. If we take $L = H_n$, then the equality is attained. □

Similar variational characterizations can be written for negative eigenvalues (one can simply pass to $-A$).

Note that in the infinite-dimensional case it is necessary to separate positive and negative eigenvalues. In the finite-dimensional case the stated equalities are fulfilled for all eigenvalues written in the order of decreasing.

The next rather non-obvious result is clear from our discussion.

7.5.4. Corollary. *Let A and B be compact selfadjoint operators on a real or complex Hilbert space H such that $(Ax, x) \leqslant (Bx, x)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences of their positive eigenvalues written in the order of decreasing taking into account their multiplicities. Then for every n we have $\alpha_n \leqslant \beta_n$.*

7.6. Unitary Operators

Let H be a complex Hilbert space and let A be a bounded operator on H. We recall that $\sigma(A^*)$ is the set complex-conjugate to $\sigma(A)$ (Remark 7.1.10).

7.6.1. Definition. *A linear operator U on a Hilbert space $H \neq 0$ is called unitary if it maps H onto H and preserves the inner product, i.e.,*

$$(Ux, Uy) = (x, y) \quad \text{for all } x, y \in H.$$

A unitary isomorphism of nonzero Hilbert spaces H_1 and H_2 is a one-to-one linear operator $U\colon H_1 \to H_2$ preserving the inner product.

An equivalent definition of a unitary operator is the equality

$$U^{-1} = U^*$$

or two equalities

$$UU^* = U^*U = I.$$

Note that to ensure that the operator U is unitary it is not enough to have only one of these equalities if we do not require its invertibility. For example, let $Ux = (0, x_1, x_2, \ldots)$ on l^2. Then U preserves the inner product and $U^*U = I$, but $UU^* \neq I$.

7.6.2. Example. Let A_φ be the operator of multiplication on $L^2(\mu)$ by a bounded μ-measurable function φ considered in Example 7.1.11. Then the operator A_φ is unitary if and only if $|\varphi(\omega)| = 1$ for μ-a.e. ω. Indeed, the equality $A_\varphi A_\varphi^* = I$ gives the equality $\varphi(\omega)\overline{\varphi(\omega)} = 1$ for μ-a.e. ω. Conversely, the latter equality yields that $A_\varphi A_\varphi^* = I = A_\varphi^* A_\varphi$.

7.6.3. Lemma. *The spectrum of a unitary operator belongs to the unit circumference.*

PROOF. Let U be a unitary operator. Then $\|U\| = 1$ and hence the spectrum of U is contained in the unit disc. The same is true for U^*, but $U^* = U^{-1}$, which excludes from the spectrum all inner points of the disc: if the operator $U - \lambda I$ is not invertible, then so is $I - \lambda U^*$, hence also $\lambda^{-1} I - U^*$. $\square$

7.6.4. Definition. *An operator A_1 on a Hilbert space H_1 is called unitarily equivalent to an operator A_2 on a Hilbert space H_2 if there exists a unitary isomorphism $J\colon\ H_1 \to H_2$ such that $A_1 = J^{-1}A_2 J$, i.e., we have a commutative diagram*

$$\begin{array}{ccc} H_1 & \xrightarrow{A_1} & H_1 \\ \Big\downarrow{\scriptstyle J} & & \Big\uparrow{\scriptstyle J^{-1}} \\ H_2 & \xrightarrow{A_2} & H_2 \end{array}$$

7.6.5. Lemma. *A linear isometry of Hilbert spaces is a unitary isomorphism.*

PROOF. Let $J\colon\ H_1 \to H_2$ be a linear isometry. Then $\|Jx+Jy\|^2 = \|x+y\|^2$, $\|Jx\|^2 = \|x\|^2$, $\|Jy\|^2 = \|y\|^2$, whence $(Jx, Jy) + (Jy, Jx) = (x, y) + (y, x)$, i.e., $\mathrm{Re}(Jx, Jy) = \mathrm{Re}(x, y)$. Replacing x by ix in the complex case, we obtain the equality for the imaginary parts. Any isometry is surjective by definition. $\square$

Characteristics of an operator that do not change under unitary equivalence are called *unitary invariants*. For example, the spectrum and norm are unitary invariants.

7.6.6. Lemma. *Let H_1 and H_2 be Hilbert spaces, $H_0 \subset H_1$ an everywhere dense linear subspace, and $U\colon\ H_0 \to H_2$ a linear mapping preserving the inner product. Then U uniquely extends to a linear mapping from H_1 to H_2 preserving the inner product and having a closed range. If $U(H_0) \neq 0$ is dense in H_2, then the extension is a unitary isomorphism.*

PROOF. Let $x \in H_1$. Let us take $x_n \in H_0$ with $x_n \to x$. Then the sequence $\{Ux_n\}$ is Cauchy in H_2 and hence converges to some element $y \in H_2$. Set $Ux := y$. It is clear from our assumption that y is independent of our choice of a sequence converging to x. If $z_n \to z$, then

$$\alpha x_n + \beta z_n \to \alpha x + \beta z \quad \text{and} \quad \alpha U x_n + \beta U z_n \to \alpha U x + \beta U z,$$

whence $U(\alpha x + \beta z) = \alpha U x + \beta U z$. Thus, the extension is linear. In addition,

$$(Ux, Uz) = \lim_{n\to\infty} (Ux_n, Uz_n) = \lim_{n\to\infty} (x_n, z_n) = (x, z).$$

Since U preserves the norm, the set $U(H_1)$ is closed. If it is dense, then it coincides with H_2. $\square$

A generalization of a unitary operator is a *partial isometry*. This is an operator J that is defined on a closed linear subspace H_1 in a Hilbert space H and maps it with the preservation of norm onto a closed subspace $H_2 \subset H$. It is clear that such an operator preserves also the inner product. However, it does not always extend to a unitary operator. For example, the operator $J\colon\ (0, x_1, x_2, \ldots) \mapsto (x_1, x_2, \ldots)$ isometrically maps a closed hyperplane in l^2 onto the whole space and cannot be extended to an isometry operator on all of l^2. Every partial isometry V has a maximal partially isometric extension $\widetilde{V}$. Indeed, if H_1 does not coincide with H and $V(H_1) \neq H$, then we take the orthogonal complements $E_1 := H_1^{\perp}$ and $E_2 := V(H_1)^{\perp}$. The following cases are possible: 1) the spaces E_1 and E_2 are isometric and hence V can be extended to a unitary isomorphism, 2) E_2 is larger than E_1, i.e., E_2 is not isometric to E_1, but has a closed subspace E_2' isometric to E_1, which gives an isometry between H and $V(H_1) \oplus E_2'$, 3) E_1 is larger than E_2, i.e., E_1 is not isometric to E_2, but has a closed subspace E_1' isometric to E_2, which gives an isometry between $H_1 \oplus E_1'$ and H.

Let us note some properties of partial isometries. We recall that an orthogonal projection is an operator of the orthogonal projecting onto a closed subspace (see Corollary 5.4.6).

7.6.7. Proposition. *Let $V \in \mathcal{L}(H)$. The following conditions are equivalent*:

(i) *the operator V is a partial isometry on the orthogonal complement of its kernel*;

(ii) *V^*V is an orthogonal projection onto some closed subspace*;

(iii) $V = VV^*V$.

In this case $\operatorname{Ker} V^{\perp} = V^*V(H)$ *and V^* is also a partial isometry from $V(H)$ onto* $\operatorname{Ker} V^{\perp}$ *vanishing on $V(H)^{\perp}$.*

PROOF. If V isometrically maps H_1 onto $V(H_1)$ and vanishes on $H_1^{\perp}$, then V^* vanishes on $V(H_1)^{\perp}$ and isometrically maps $V(H_1)$ onto H_1. Indeed, the equality $(Vx, y) = (x, V^*y)$ yields that $V^*y = 0$ if $y \perp V(H_1) = V(H)$. Whenever $y \in V(H_1)$, i.e., $y = Vz$ with $z \in H_1$, we have $V^*y \perp \operatorname{Ker} V$, i.e., $V^*y \in H_1$. Then $(Vx, y) = (x, z) = (x, V^*y)$ for all $x \in H_1$, whence $V^*y = z$. Hence V^*V is the projection onto H_1, whence $V = VV^*V$.

Let V^*V be the orthogonal projection P onto a closed subspace H_1. Then $V = 0$ on $H_1^{\perp}$ and V is an isometry on H_1, since

$$(Vx, Vx) = (V^*Vx, x) = (Px, x) = (Px, Px).$$

It is clear that we also have the equality $V = VV^*V$. Finally, suppose that V satisfies the latter equality. Then $V^*V = (V^*V)^2$, i.e., the selfadjoint operator $A = V^*V$ satisfies the identity $A = A^2$. It is readily seen (this is done below in Lemma 7.9.1) that A is an orthogonal projection. □

It follows that an operator V is a maximal partial isometry defined by zero on the orthogonal complement of the subspace on which it is an isometry if and only if either $V^*V = I$ or $VV^* = I$ (i.e., one of the operators V or V^* is isometric on all of H).

Extensions of isometric operators will be considered in Chapter 10 in connection with extensions of symmetric (unbounded) operators. In the next section partial isometries are used for obtaining the so-called polar decompositions of operators.

7.7. Continuous Functions of Selfadjoint Operators

Any linear operator A on a space X can be substituted as an argument of a polynomial f of one variable to obtain the operator $f(A)$. Here we establish a highly nontrivial fact that selfadjoint operators can be substituted in arbitrary continuous functions of one variable.

7.7.1. Lemma. *Let A be a selfadjoint operator on a nonzero complex Hilbert space and let P be a polynomial with complex coefficients. Then*

$$\|P(A)\| = \max_{t\in\sigma(A)} |P(t)| \leqslant \max_{t\in[-\|A\|,\|A\|]} |P(t)|. \tag{7.7.1}$$

PROOF. Let $P(t) = \sum_{k=0}^n c_k t^k$. Then

$$P(A)^*P(A) = \sum_{k=0}^n \overline{c_k} A^k \sum_{k=0}^n c_k A^k = Q(A),$$

$Q\colon t\mapsto \overline{P(t)}P(t)$ is a polynomial, $P(A)^*P(A)$ is selfadjoint. Hence

$$\begin{aligned}\|P(A)\|^2 &= \sup_{\|x\|\leqslant 1} \bigl(P(A)x, P(A)x\bigr)\\ &= \sup_{\|x\|\leqslant 1} \bigl(P(A)^*P(A)x, x\bigr) = \|P(A)^*P(A)\|\\ &= \sup_{\lambda\in\sigma(P(A)^*P(A))} |\lambda| = \sup_{t\in\sigma(A)} |\overline{P(t)}P(t)| = \sup_{t\in\sigma(A)} |P(t)|^2,\end{aligned}$$

where the third and forth equalities follow from Theorem 7.2.6, and the last but one equality is obtained from Theorem 7.1.9. □

In the real case the same is true for real polynomials.

With the aid of this lemma it is easy to define continuous functions of a selfadjoint operator. We recall that an *algebra* is a linear space $\mathcal{L}$ equipped with an associative multiplication $(a,b)\mapsto ab$ for which $(\lambda a)b = a(\lambda b) = \lambda ab$, $(a+b)(c+d) = ac+bc+ad+bd$ for all $a,b,c,d\in\mathcal{L}$ and all scalars λ (see Chapter 11). The most important examples for us is the algebra $\mathcal{L}(H)$ of bounded operators on a Hilbert space H and the algebra $C(K)$ of continuous complex functions on a compact space K. A homomorphism of an algebra is a linear operator J with $J(ab) = J(a)J(b)$.

7.7.2. Theorem. *Let A be a selfadjoint operator on a complex Hilbert space $H\neq 0$. There is a unique homomorphism J of the algebra $C\bigl(\sigma(A)\bigr)$ to the algebra $\mathcal{L}(H)$ such that*

1) $J(P) = P(A)$ *for every polynomial* $P\colon \mathbb{R}^1\to\mathbb{C}$,

2) $\|J(f)\| = \sup_{t\in\sigma(A)} |f(t)|$ *for all* $f\in C\bigl(\sigma(A)\bigr)$,

3) $J(f)^* = J(\overline{f})$ *for all* $f\in C\bigl(\sigma(A)\bigr)$.

A similar assertion is true in case of real spaces H and $C\bigl(\sigma(A)\bigr)$.

PROOF. For every polynomial f we set

$$J(f) := f(A).$$

For every $f \in C\big(\sigma(A)\big)$ there exists a sequence of polynomials f_n uniformly converging to f on the compact set $\sigma(A)$ on the real line. By the lemma the sequence of operators $f_n(A)$ is Cauchy in $\mathcal{L}(H)$ and hence converges in the operator norm to some operator $J(f) \in \mathcal{L}(H)$. It is important that this operator does not depend on the approximating sequence: if polynomials g_n also converge uniformly to f on $\sigma(A)$, then the polynomials $f_1, g_1, f_2, g_2, \ldots$ converge as well, which proves our assertion. If polynomials φ_n converge in $C\big(\sigma(A)\big)$ to a function φ and polynomials ψ_n converge to a function ψ, then $\varphi(A)\psi(A)$ equals

$$\lim_{n\to\infty} \varphi_n(A) \lim_{n\to\infty} \psi_n(A) = \lim_{n\to\infty} \varphi_n(A)\psi_n(A) = (\varphi \cdot \psi)(A),$$

since the polynomials $\varphi_n\psi_n$ converge to the function $\varphi\psi$. Thus, we have constructed a homomorphism. Moreover, $\|f(A)\| = \lim\limits_{n\to\infty} \|f_n(A)\|$, which proves 2). In addition, $J(\overline{f}) = J(f)^*$. The proof also shows the uniqueness of a homomorphism with the indicated properties. □

Letting $f(A) := J(f)$ for each continuous function f on the whole real line, we obtain the equality $\|f(A)\| = \sup\limits_{t\in\sigma(A)} |f(t)|$.

7.7.3. Corollary. *Let $f \in C\big(\sigma(A)\big)$ and $f(t) \geqslant 0$ for all $t \in \sigma(A)$. Then the operator $f(A)$ is selfadjoint and $f(A) \geqslant 0$.*

PROOF. The function $\sqrt{f} \geqslant 0$ is continuous on $\sigma(A)$. Hence the operator $B = \sqrt{f}(A)$ is selfadjoint. Since we have $f(A) = B^2$ and $(Ax, x) = (Bx, Bx) \geqslant 0$, the operator $f(A)$ is selfadjoint and nonnegative. □

Taking the function $f(t) = \sqrt{t}$ on $[0, +\infty)$ in case of an operator $A \geqslant 0$, we obtain the operator $\sqrt{A}$.

7.7.4. Corollary. *If $A \geqslant 0$, then the operator $\sqrt{A}$ is selfadjoint, nonnegative and $A = \sqrt{A}\sqrt{A}$.*

Note that the operator $\sqrt{A}$ is the only nonnegative operator the square of which equals A (Exercise 7.10.73).

7.7.5. Corollary. *For every bounded operator A on a Hilbert space H, the operator*

$$|A| := (A^*A)^{1/2}$$

is well-defined and nonnegative. The operator $|A|$ is called the absolute value of A.

PROOF. We have $(A^*Ax, x) = (Ax, Ax) \geqslant 0$. □

7.7.6. Example. Let A_φ be the operator of multiplication on $L^2(\mu)$ by a bounded μ-measurable function φ considered in Example 7.1.11. The operator A_φ can be substituted in an arbitrary bounded Borel function f on the complex plane (not necessarily continuous), by defining $f(A_\varphi)$ as the operator $A_{f\circ\varphi}$ of multiplication by the bounded μ-measurable function $f\circ\varphi$. Clearly, for any polynomial f this gives the operator $f(A_\varphi)$. Moreover, it is easy to see from Theorem 7.7.2 that in case of a real function φ the operator $f(A_\varphi)$ is the operator of multiplication by the function $f\circ\varphi$. Below, when we establish a unitary equivalence of every selfadjoint operator to some operator of multiplication A_φ, this will enable us to define easily Borel functions of selfadjoint operators.

The next useful result represents general operators by means of selfadjoint operators and partial isometries and is called the *polar decomposition* of an operator.

7.7.7. Theorem. *Let A be a bounded operator on a complex or real Hilbert space H. Then there exists a partial isometry U on the closure of the range of $|A|$, equal the orthogonal complement of the kernel of A, for which*

$$A = U|A|.$$

If $A \neq 0$ has a zero kernel and dense range, then U is a unitary operator.

PROOF. Set

$$Uy := Ax, \quad y = |A|x \in |A|(H).$$

Since

$$(Ax, Ax) = (A^*Ax, x) = (|A|^2x, x) = (|A|x, |A|x),$$

whenever $|A|x = 0$ we have $Ax = 0$, which proves that U is well-defined. From the equalities above we obtain $(Uy, Uy) = (y, y)$, i.e., U is an isometry on $|A|(H)$. Hence U extends to a partial isometry on the closure of $|A|(H)$. The same equalities show that $\operatorname{Ker} A = \operatorname{Ker}|A|$. Since $|A|$ is a selfadjoint operator, the closure of its range is the orthogonal complement of the kernel (see Lemma 6.8.4). If the operator A is injective and has a dense range, then the operator U on the closure of $|A|(H)$ is everywhere defined and has a dense range. Hence it is unitary. □

Usually it is convenient to extend U to an operator on the whole space H by setting $U|_{\operatorname{Ker} A} = 0$. Below, when it becomes necessary to consider U on all of H, we shall have in mind this extension. The way of extending does not influence $U|A|$, however, the chosen extension makes the adjoint operator U^* also a partial isometry. Indeed, the extended operator U is zero on $\operatorname{Ker} A$ and linearly and isometrically maps the closed subspace

$$H_1 := (\operatorname{Ker} A)^\perp = \overline{A^*(H)}$$

onto the closed subspace

$$H_2 := \overline{A(H)} = (\operatorname{Ker} A^*)^\perp.$$

The orthogonal decomposition

$$H = \operatorname{Ker} A \oplus \overline{A^*(H)} = \operatorname{Ker} A^* \oplus \overline{A(H)}$$

shows that the operator U^* is zero on $\operatorname{Ker} A^*$ and maps H_2 isometrically onto H_1 by means of the inverse to $U|_{H_1}$ (here we could also refer to Proposition 7.6.7). Hence U^*U is the orthogonal projection onto H_1, and UU^* is the orthogonal projection onto H_2. This gives (under the indicated extension of U, of course) the equality

$$|A| = U^*A.$$

Concerning the uniqueness of the polar decomposition, see Exercise 7.10.86.

The polar decomposition resembles the representation of a complex number z in the form $z = e^{i\theta}|z|$. This analogy is limited, though. Say, it is not always true that $|A+B| \leqslant |A|+|B|$. In addition, one does not always have $|A| = |A^*|$. For example, if the operator A on l^2 is the right shift $(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$, then $A^*A = I$ and $|A| = I$, but $AA^*x = (0, x_2, x_3, \ldots)$ and $|A^*|^2 = AA^* \neq I$.

If the operator U is unitary (i.e., A is injective and has a dense range), then $W = U^*$ is also unitary, so $A^* = |A|W$, where W is unitary. Since A^* is also injective and has a dense range, we have similarly $A = |A^*|V$, where V is unitary. However, in the general case it is not always possible to decompose A into the product $A = ST$ of a selfadjoint operator S and a unitary operator T. As an example take the shift considered above: the operator S would have the zero kernel and a range that is not dense, which is impossible for a selfadjoint operator.

By using the polar decomposition and the Hilbert–Schmidt theorem we obtain the following representation of an arbitrary compact operator on a Hilbert space.

7.7.8. Proposition. *Let K be a compact operator on a complex or real separable Hilbert space $H \neq 0$. Then there exist two orthonormal sequences $\{\varphi_n\}$ and $\{\psi_n\}$ and a sequence of real numbers λ_n tending to zero for which*

$$Kx = \sum_{n=1}^{\infty} \lambda_n (x, \varphi_n)\psi_n, \quad x \in H.$$

In addition, every operator of the indicated form is compact.

PROOF. Let $K = U|K|$ be the polar decomposition. Let $\{\varphi_n\}$ be an eigenbasis of the selfadjoint compact operator $|K|$ and $|K|\varphi_n = \lambda_n\varphi_n$. Set $\psi_n := U\varphi_n$ for all n such that $U\varphi_n \neq 0$. Since $|K|x = \sum_{n=1}^{\infty} \lambda_n(x, \varphi_n)\varphi_n$, applying U to both parts of this equality we arrive at the desired representation (of course, excluding numbers n with $U\varphi_n = 0$ we must renumber φ_n and ψ_n). □

7.8. The Functional Model

The main result of this section shows that up to an isomorphism any selfadjoint operator is the operator of multiplication by a real function. This assertion is a continual analog of the fact known from linear algebra about representing a symmetric matrix in a diagonal form. The nontriviality of the generalization is, in particular, that in the infinite-dimensional case a selfadjoint operator can fail to have eigenvectors. The representation of an operator in the form of a multiplication by a function is called the functional model of this operator. In the next section we discuss a representation of a selfadjoint operator in the form of an integral with respect to a projection-valued measure. First we consider operators analogous to finite-dimensional operators without multiple eigenvalues. This analog is the following concept.

7.8.1. Definition. *We shall say that an operator A on a normed space X has a cyclic vector h if the linear span of the vectors $h, Ah, A^2h, \ldots$ is everywhere dense in X.*

7.8.2. Example. The unit operator on any space of dimension greater than 1 has no cyclic vectors. The operator A_φ of multiplication by the argument on $L^2(\mu)$, where μ is a bounded Borel measure on an interval, has cyclic vectors. For example, for a cyclic vector we can take the function $h(t) = 1$, because the linear span of the functions $1, t, t^2, \ldots$ consists of all polynomials and hence is dense in the space $L^2(\mu)$.

We shall say that a Hilbert space H is an *orthogonal sum* of its closed subspaces H_n if $H_n \perp H_k$ whenever $n \neq k$ and every vector $h \in H$ is the sum of the series $\sum_{n=1}^\infty P_n h$, where P_n is the operator of orthogonal projection onto H_n.

7.8.3. Lemma. *Let A be a selfadjoint operator on a separable Hilbert space H. Then H is the orthogonal sum of closed subspaces H_n such that $A(H_n) \subset H_n$ and $A|_{H_n}$ possesses a cyclic vector.*

PROOF. Let us consider the family $\mathcal{S}$ the elements of which are collections of pairwise orthogonal closed subspaces $E \subset H$ such that $A(E) \subset E$ and $A|_E$ has a cyclic vector. This family is partially ordered by inclusion. Every chain in $\mathcal{S}$ has a majorant: the union of its elements. Hence $\mathcal{S}$ contains a maximal element S. By the separability of H this element S consists of a finite or countable collection of pairwise orthogonal subspaces H_n with the indicated property. The linear span of all H_n is everywhere dense in H, because otherwise we can find a nonzero vector $h \perp H_n$ for all n. Then the closure E of the linear span of the sequence of vectors $A^k h$, $k = 0, 1, \ldots$, is orthogonal to all H_n and is mapped by A into itself, while the vector h is cyclic for $A|_E$. However, this is impossible by the maximality of S. It remains to observe that $h = \sum_{n=1}^\infty P_n h$ for every $h \in H$ (P_n is the projection onto H_n), since otherwise the vector $a = h - \sum_{n=1}^\infty P_n h$ is orthogonal to all H_n and cannot be approximated by vectors from the linear span of H_n. $\square$

7.8.4. Remark. Similarly one can define a decomposition $H = \bigoplus_\gamma H_\gamma$ with an uncountable number of pairwise orthogonal separable closed subspaces H_γ. Then the previous lemma remains in force also in the case of nonseparable H, which is proved by the same reasoning.

7.8.5. Theorem. *Let A be a selfadjoint operator with a cyclic vector on a separable Hilbert space $H \neq 0$. Then there exists a nonnegative Borel measure μ on the compact set $\sigma(A)$ such that the operator A is unitarily equivalent to the operator of multiplication by the argument in $L^2(\mu)$, i.e., to the operator A_φ with the function $\varphi(t) = t$.*

PROOF. Let h be a cyclic vector for A. We define a functional l on $C\bigl(\sigma(A)\bigr)$ by the formula $l(f) = \bigl(f(A)h, h\bigr)$. Clearly, we have obtained a continuous linear functional. If $f \geqslant 0$, then $l(f) \geqslant 0$ according to the results in the previous section. By the Riesz theorem there exists a nonnegative Borel measure μ on the compact set $\sigma(A)$ for which

$$l(f) = \int_{\sigma(A)} f(t)\, \mu(dt), \quad f \in C\bigl(\sigma(A)\bigr).$$

For any nonnegative integer k we set

$$U(A^k h) := p_k, \quad \text{where } p_k(t) = t^k.$$

We extend U by linearity to finite linear combinations of vectors $A^k h$. Since

$$\begin{aligned}\Big(\sum \alpha_k A^k h, \sum \beta_m A^m h\Big) &= \Big(\Big(\sum \overline{\beta_m} A^m\Big)\Big(\sum \alpha_k A^k\Big)h, h\Big) \\ &= \int_{\sigma(A)} \Big(\sum \overline{\beta_m} t^m\Big)\Big(\sum \alpha_k t^k\Big)\,\mu(dt) \\ &= \Big(\sum \alpha_k p_k, \sum \beta_m p_m\Big)_{L^2(\mu)},\end{aligned}$$

the mapping U is well-defined when extended by linearity (i.e., equal linear combinations are taken to the same element of $L^2(\mu)$) and preserves the inner product. The range of U is everywhere dense in $L^2(\mu)$, because the set of polynomials (the linear span of $\{p_k\}$) is everywhere dense in $L^2(\mu)$. By Lemma 7.6.6 we can extend U to a unitary operator from H to $L^2(\mu)$. By construction we have

$$\begin{aligned}UA\Big(\sum \alpha_k A^k h\Big) &= U\Big(\sum \alpha_k A^{k+1} h\Big) \\ &= \sum \alpha_k p_{k+1} = A_\varphi \sum \alpha_k p_k = A_\varphi U\Big(\sum \alpha_k A^k h\Big),\end{aligned}$$

where $\varphi(t) = t$. Thus, $UA = A_\varphi U$, i.e., $A_\varphi = UAU^{-1}$. □

Let us now consider the general case of the *spectral theorem*.

7.8.6. Theorem. (THE FUNCTIONAL MODEL OF A SELFADJOINT OPERATOR) *Let A be a selfadjoint operator on a separable Hilbert space $H \neq 0$. Then there exists a finite nonnegative Borel measure μ along with a bounded Borel function φ on the real line such that the operator A is unitarily equivalent to the operator of multiplication by φ in $L^2(\mu)$. Moreover, the function φ can be taken with values in $\sigma(A)$.*

PROOF. Let us decompose H in the orthogonal sum of closed subspaces H_n invariant under A such that the operators $A|_{H_n}$ have cyclic vectors. The orthogonal projection of h onto H_n is denoted by h_n. For every n, we find a probability Borel measure μ_n on the compact set $\sigma(A) \subset [-\|A\|, \|A\|]$ for which the operator $A|_{H_n}$ is unitarily equivalent to multiplication by the argument in $L^2(\mu_n)$. We can assume that $\|A\| < 1$, i.e., $\sigma(A) \subset [a,b] \subset (-1,1)$. Let us translate the measure μ_n to the interval $\Omega_n = (2n-3, 2n-1)$ and denote the obtained measure by ν_n (it is concentrated on $\sigma(A) + 2n - 2$). Clearly, the operator $A|_{H_n}$ is unitarily equivalent to the operator of multiplication by the function $t \mapsto t - 2n + 2$ in $L^2(\nu_n)$. Let $J_n\colon H_n \to L^2(\nu_n)$ be the corresponding unitary equivalence. Let

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \nu_n$$

and $\varphi(t) = t - 2n + 2$ if $t \in \sigma(A) + 2n - 2$. Outside these compact sets we define φ in an arbitrary way to make it a Borel function; for example, we can make it equal to a fixed number in $\sigma(A)$, which gives a function with values in $\sigma(A)$, but it can also be made a continuous periodic piece-wise linear function with $\varphi(t) = t$ if $t \in [a, b]$ (however, it is not always possible to obtain a continuous function with values in $\sigma(A)$). On the space $L^2(\mu)$ we have the operator A_φ of multiplication by φ. We now define an operator $J\colon\ H \to L^2(\mu)$ by the equality

$$J\colon\ h = \sum_{n=1}^{\infty} h_n \mapsto \sum_{n=1}^{\infty} 2^{n/2} J_n h_n.$$

The series in the right-hand side converges in $L^2(\mu)$, since the functions $J_n h_n$ have supports in pairwise disjoint sets $\sigma(A) + 2n - 2$ and the integral of a function with respect to the measure μ equals the sum of its integrals with respect to the measures $2^{-n}\nu_n$. Let us verify that we have obtained the desired objects. For every $h \in H$ we have

$$\|Jh\|^2_{L^2(\mu)} = \sum_{n=1}^{\infty} 2^{-n} 2^n \|J_n h_n\|^2_{L^2(\nu_n)} = \sum_{n=1}^{\infty} \|h_n\|^2 = \|h\|^2.$$

In addition,

$$A_\varphi J h = \sum_{n=1}^{\infty} \varphi 2^{n/2} J_n h_n = \sum_{n=1}^{\infty} 2^{n/2} J_n A h_n = JAh.$$

Thus, J is a unitary equivalence of the operators A and A_φ. □

7.8.7. Remark. (i) In the proof we have constructed a measure on the real line and the function φ has been taken either with values in $\sigma(A)$ or continuous and piece-wise linear, but it is easy to modify our construction in order to obtain a measure on an interval (it is also possible to transform the real line into an interval by using arctg). A continuous version of φ with values in the spectrum of A can fail to exist (say, if the spectrum is not an interval).

(ii) Remark 7.8.4 enables us to obtain an analog of this theorem in case of a nonseparable Hilbert space. To this end, we represent H as the orthogonal sum of pairwise orthogonal separable closed subspaces H_γ, $\gamma \in \Gamma$, invariant with respect to A. The restriction of A to H_γ is unitarily equivalent to the multiplication by a measurable function $\varphi_\gamma\colon\ \mathbb{R}^1 \to \sigma(A)$ in $L^2(\mu_\gamma)$, where μ_γ is some Borel measure on the real line. Now we can take Γ disjoint copies of the real line and take for μ the sum of the measures μ_γ on these copies (which gives a countably additive measure with values in $[0, +\infty]$; such a measure need not be σ-finite).

(iii) The choice of H_n is not unique. For example, for the operator of multiplication by the argument on $L^2[-1, 1]$ we can take the subspace H_1 of functions equal to zero on $(0, 1]$ and the subspace H_2 of functions equal to zero on $[-1, 0)$. The operators $A|_{H_1}$ and $A|_{H_2}$ have cyclic vectors. The same is possible when there is no cyclic vector in H. In §10.4 we discuss some canonical decompositions invariant under unitary isomorphisms. Note also that the spectrum of the restriction of A to H_n can be strictly smaller than $\sigma(A)$.

Representations of an operator as the multiplication by a function enable us to substitute it into Borel functions.

7.8.8. Definition. *Let A be a selfadjoint operator on a separable Hilbert space $H \neq 0$ and let f be a bounded Borel function on the real line. We know that A is unitarily equivalent by means of an isomorphism J to the operator of multiplication by a bounded Borel function φ in $L^2(\mu)$. Let us define the operator $f(A)$ by the formula*

$$f(A) := J^{-1} A_{f\circ\varphi} J.$$

This definition extends to the nonseparable case by using a decomposition of the space on which the given selfadjoint operator acts into the direct sum of invariant separable subspaces. It is readily seen that this definition agrees with the earlier introduced construction of a continuous function of a selfadjoint operator. In the next section we represent Borel functions of a selfadjoint operator by integrals with respect to projection-valued measures, whence it will follow that our definition of Borel functions of a selfadjoint operator does not depend on our choice of the functional model.

7.8.9. Corollary. *To every bounded complex Borel function f on the real line this definition associates the operator $f(A) \in \mathcal{L}(H)$ with the following properties: if $f(t) = g(t)$ for $t \in \sigma(A)$, then $f(A) = g(A)$, and $f_n(A)x \to f(A)x$ for all vectors $x \in H$ if $f_n(t) \to f(t)$ for all points $t \in \sigma(A)$ and $|f_n(t)| \leqslant C < \infty$ for all $n \in \mathbb{N}$ and $t \in \sigma(A)$.*

PROOF. Let us write A as the operator of multiplication by a function φ with values in $\sigma(A)$. Then the first assertion is obvious and the second one follows from the Lebesgue dominated convergence theorem. □

On account of Example 7.1.11 we obtain the following assertion.

7.8.10. Corollary. *Let A be a selfadjoint operator and let f be a continuous complex function on $\sigma(A)$. Then*

$$\sigma\bigl(f(A)\bigr) = f\bigl(\sigma(A)\bigr).$$

PROOF. We can assume that A is the operator of multiplication by a bounded Borel function φ on $L^2(\mu)$, where μ is a bounded nonnegative Borel measure on the real line. For μ-a.e. t, the number $\varphi(t)$ belongs to the spectrum of A. Then $f(A)$ is the multiplication by the function $f\circ\varphi$, which is defined μ-almost everywhere. If λ is an essential value of φ, then $f(\lambda)$ is an essential value of $f\circ\varphi$, since the set $f^{-1}\bigl(\{z\colon\ |f(\lambda) - z| < r\}\bigr)$ contains a neighborhood of λ for every $r > 0$. If the point η does not belong to the compact set $f\bigl(\sigma(A)\bigr)$, then it is at some positive distance ε from this set, hence $\bigl|f\bigl(\varphi(t)\bigr) - \eta\bigr| \geqslant \varepsilon$ for μ-a.e. t, which gives the invertibility of the operator $f(A) - \eta I$. □

An important example of a function of a selfadjoint operator A is its *Caley transform*

$$U = (A - iI)(A + iI)^{-1} = \varphi(A),$$

where $\varphi(t) = (t - i)/(t + i)$. Since $|\varphi(t)| = 1$, the operator U is unitary. The operator A is reconstructed by U by the formula

$$A = i(I + U)(I - U)^{-1},$$

since the spectrum of U does not contain 1 by the previous corollary. Conversely, for every unitary operator U the spectrum of which does not contain 1 the operator $i(I+U)(I-U)^{-1}$ is selfadjoint, which is readily verified. Indeed, the adjoint of this operator is $-i(I+U^*)(I-U^*)^{-1}$. In addition,

$$-(I+U^*)(I-U^*)^{-1} = (I+U)(I-U)^{-1},$$

since multiplying both sides by the invertible operator $(I-U^*)(I-U)$ and using the fact that the operators U and U^* commute, we arrive at the obvious equality $-(I+U^*)(I-U) = (I+U)(I-U^*)$ (on both sides we have $U-U^*$).

If the operator A has no cyclic vectors, it cannot be unitarily equivalent to the operator of multiplication by the argument on $L^2(\mu)$ for a measure μ on an interval, since this operator of multiplication possesses a cyclic vector and the property to have cyclic vectors is obviously preserved by unitary isomorphisms. However, even in the case of the absence of cyclic vectors any selfadjoint operator is represented in the form of multiplication by the argument in the space of vector functions (see § 7.10(viii)).

7.8.11. Remark. It is instructive to compare the Hilbert–Schmidt theorem about diagonalization of a compact selfadjoint operator A on a separable Hilbert space H with the theorem about representation of A in the form of multiplication by a bounded real measurable function φ on $L^2(\mu)$, where μ is a bounded Borel measure on the real line. If the operator A has no multiple eigenvalues (including the zero eigenvalue) and the space H is infinite-dimensional, then A is unitarily equivalent to the operator of multiplication by the argument on $L^2(\mu)$, where μ is a probability measure concentrated on the set $\{\alpha_n\}$ of all eigenvalues of A; we can assume that the value of μ at α_n is 2^{-n}. In case of multiple eigenvalues one has to decompose A in a direct sum of operators without multiple eigenvalues and apply the same construction as in the general theorem. On the other hand, the Hilbert–Schmidt theorem can be derived from the general spectral theorem. For this it suffices to verify that if the operator of multiplication by φ on $L^2(\mu)$ is compact, then the restriction of the measure μ to the set $\{t\colon\ \varphi(t)\neq 0\}$ is concentrated at countably many points (then their indicator functions will be eigenfunctions). This verification is Exercise 7.10.87.

7.8.12. Proposition. *Let A be a bounded operator on a Hilbert space H.*

(i) *The operator A is not compact precisely when there exists an infinite-dimensional closed subspace $H_0\subset H$ such that the restriction of A to H_0 has a bounded inverse operator, i.e., the mapping $A\colon\ H_0\to A(H_0)$ is one-to-one and the inverse is continuous.*

(ii) *The operator A is compact precisely when*

$$\lim_{n\to\infty}\|Ae_n\| = 0$$

for every infinite orthonormal sequence $\{e_n\}$ in H. This is equivalent to the property that

$$\lim_{n\to\infty}(A\psi_n,\varphi_n) = 0$$

for all infinite orthonormal sequences $\{\psi_n\}$ and $\{\varphi_n\}$ in the space H (*see, however, Exercise* 7.10.114).

PROOF. (i) We know that a compact operator cannot be invertible on an infinite-dimensional space. Suppose that A is not compact. It suffices to consider separable H. We first consider the case where A is selfadjoint and nonnegative. We can assume that A is the multiplication by a bounded nonnegative Borel function φ on $L^2(\mu)$, where μ is a Borel probability measure on the real line. Then for some $\varepsilon > 0$ the Borel set $E_\varepsilon := \{t\colon\ \varphi(t) \geqslant \varepsilon\}$ possesses the property that the subspace H_0 in $L^2(\mu)$ consisting of functions equal to zero outside E_ε is infinite-dimensional. In the opposite case the operator A would be compact, because it is the limit (in the operator norm) of the operators of multiplication by $\varphi I_{E_\varepsilon}$ as $\varepsilon \to 0$. It is clear that on H_0 our operator is invertible.

In the general case we take the polar decomposition $A = U|A|$. Then $|A|$ is not compact. As we have shown, there is an infinite-dimensional closed subspace H_0 on which $|A|$ is invertible, i.e., there is $c > 0$ such that $\|x\| \leqslant c\||A|x\|$ whenever $x \in H_0$. Then we have $\|x\| \leqslant c\|Ax\|$ for all $x \in H_0$.

(ii) Let $A \in \mathcal{K}(H)$ and let $\{e_n\}$ be an orthonormal sequence. Then $Ae_n \to 0$ in the weak topology, since for every $y \in H$ we have $(Ae_n, y) = (e_n, A^*y) \to 0$. Hence $\|Ae_n\| \to 0$. If $A \notin \mathcal{K}(H)$, then by (i) there exists an infinite orthonormal sequence $\{e_n\}$ with $\|Ae_n\| \geqslant c > 0$. One can also find two orthonormal sequences $\{\psi_n\}$ and $\{\varphi_n\}$ with $|(A\psi_n, \varphi_n)| \geqslant c > 0$. For this we observe that the operator $\sqrt{|A|}$ is not compact as well. Hence we can take an infinite orthonormal sequence $\{\psi_n\}$ such that $\|\sqrt{|A|}\psi_n\| \geqslant c > 0$. Such a sequence can be picked in the subspace

$$H_1 := \operatorname{Ker} \sqrt{|A|}^{\perp} = \operatorname{Ker} |A|^{\perp} = \operatorname{Ker} A^{\perp}.$$

Set $\varphi_n = U\psi_n$. This gives an orthonormal sequence, because $\psi_n \in H_1$ and the operator U is an isometry on H_1. In addition, $|A|\psi_n \in |A|(H) \subset H_1$. We obtain

$$(A\psi_n, \varphi_n) = (U|A|\psi_n, U\psi_n) = (|A|\psi_n, \psi_n) = \bigl\|\sqrt{|A|}\psi_n\bigr\|^2 \geqslant c^2,$$

which completes the proof. $\square$

Closing this section we observe that the obtained representations of selfadjoint operators in the form of operators of multiplication by functions leave open the following question: given two operators of multiplication, how can one decide whether they are equivalent? This question will be addressed in Chapter 10.

7.9. Projections and Projection-Valued Measures

An *orthogonal projection* in a Hilbert space H is the operator of the orthogonal projecting onto a closed subspace (see Corollary 5.4.6).

7.9.1. Lemma. *A bounded operator P is an orthogonal projection precisely when $P^* = P = P^2$.*

PROOF. If P is the projection onto a closed subspace H_0, then $P^2 = P$ and $(Px, y) = (x, Py)$, i.e., $P^* = P$, which has already been noted in Example 6.8.3. Conversely, if the indicated equalities are fulfilled, then we set

$$H_0 := \operatorname{Ker}(I - P), \quad H_1 := \operatorname{Ker} P.$$

It is clear that H_0 and H_1 are closed and $H_0 \perp H_1$, since for all $x \in H_0$ and $y \in H_1$ we have $(x,y) = (Px,y) = (x,Py) = 0$. Since

$$x - Px \in \operatorname{Ker} P, \quad Px \in \operatorname{Ker}(I-P),$$

for any vector x we obtain $x = (x-Px)+Px$, where $Px \in H_0$ and $x - Px \in H_1$, i.e., H is the sum of H_0 and H_1. On the subspace H_0 the operator P coincides with the identity, on H_1 it vanishes. Hence P is the projection onto H_0. □

We now discuss a representation of a selfadjoint operator in the form of an integral with respect to a projection-valued measure.

7.9.2. Definition. *Let $(\Omega, \mathcal{B})$ be a measurable space and let H be a separable Hilbert space. A mapping Π from $\mathcal{B}$ to the space $\mathcal{P}(H)$ of orthogonal projections in H is called a projection-valued measure if, for every $a, b \in H$, the complex function*

$$\Pi_{a,b}\colon\ B \mapsto \bigl(\Pi(B)a, b\bigr)$$

is a bounded countably additive measure on $\mathcal{B}$.

Here are the simplest properties of the projection-valued measure Π:

1) the mapping Π is additive, i.e., $\Pi(B_1 \cup B_2) = \Pi(B_1) + \Pi(B_2)$ for any disjoint sets $B_1, B_2 \in \mathcal{B}$;
2) $\Pi(B_1) \leqslant \Pi(B_2)$ if $B_1, B_2 \in \mathcal{B}$ and $B_1 \subset B_2$;
3) $\Pi(B_1)\Pi(B_2) = \Pi(B_2)\Pi(B_1) = 0$ for any disjoint sets $B_1, B_2 \in \mathcal{B}$;
4) $\Pi(B_1)\Pi(B_2) = \Pi(B_2)\Pi(B_1) = \Pi(B_1 \cap B_2)$ for all $B_1, B_2 \in \mathcal{B}$.

Property 1) follows from the additivity of the measures $\Pi_{a,b}$. Property 2) follows from the equality $\Pi(B_2) = \Pi(B_1) + \Pi(B_2 \backslash B_1)$ taking into account that Π takes values in the set of nonnegative operators. For the proof of 3) we observe that 1) and the equality $\Pi(B_1 \cup B_2) = \Pi(B_1 \cup B_2)^2$ yield the equality

$$\Pi(B_1)\Pi(B_2) + \Pi(B_2)\Pi(B_1) = 0.$$

Multiplying it from the left by $\Pi(B_1)$ we obtain

$$\Pi(B_1)\Pi(B_2) + \Pi(B_1)\Pi(B_2)\Pi(B_1) = 0,$$

i.e., $\Pi(B_1)\Pi(B_2)\bigl(I+\Pi(B_1)\bigr) = 0$. Since the operator $I+\Pi(B_1) \geqslant I$ is invertible, we have $\Pi(B_1)\Pi(B_2) = 0$. The second equality in 3) follows from the first one. For any $B_1, B_2 \in \mathcal{B}$ we can write $B_1 = C_1 \cup D$, $B_2 = C_2 \cup D$, where

$$C_1 = B_1 \backslash (B_1 \cap B_2),\ C_2 = B_2 \backslash (B_1 \cap B_2),\ D = B_1 \cap B_2.$$

Since C_1, C_2, D are disjoint, the projections $\Pi(C_1)$, $\Pi(C_2)$ and $\Pi(D)$ commute as shown above and

$$\Pi(C_1)\Pi(C_2) = \Pi(C_1)\Pi(D) = \Pi(C_2)\Pi(D) = 0.$$

This gives the equality $\Pi(B_1)\Pi(B_2) = \Pi(D)^2 = \Pi(D)$.

Since each $\Pi(B)$ is a projection, we have

$$\Pi_{a,a} \geqslant 0, \quad \Pi_{a,a}(\Omega) \leqslant \|a\|^2.$$

The equality $\operatorname{Re}\Pi_{a,b} = \frac{1}{2}(\Pi_{a+b,a+b} - \Pi_{a,a} - \Pi_{b,b})$ gives the following estimate for the variation of the measure $\Pi_{a,b}$:

$$\|\Pi_{a,b}\| \leqslant 2(\|a+b\|^2 + \|a\|^2 + \|b\|^2) \leqslant 6\|a\|^2 + 6\|b\|^2.$$

For an arbitrary bounded complex $\mathcal{B}$-measurable function f, we define the integral

$$\int_\Omega f(\omega)\, d\Pi(\omega)$$

to be a bounded operator T such that

$$(Ta, b) = \int_\Omega f(\omega)\, d\Pi_{a,b}(\omega)$$

for all $a, b \in H$. Indeed, the right-hand side of this equality is linear in a, conjugate-linear in b and separately continuous in a and b, which follows from the above estimate for the variation (it is also sufficient to use the continuity of the functions $\Pi_{a,b}(B)$). One can go further and introduce the Lebesgue integral with respect to an H-valued measure in order to define Ta. Finally, one can define operator-valued integrals with the aid of partial sums converging in the operator norm. We do not develop here these approaches and give only the following assertion.

7.9.3. Proposition. *For every n, one can partition Ω into disjoint parts $\Omega_{n,1}, \ldots, \Omega_{n,n} \in \mathcal{B}$ such that for any choice of points $\omega_{n,k} \in \Omega_{n,k}$ the sums $\sum_{k=1}^n f(\omega_{n,k})\Pi(\Omega_{n,k})$ will converge to T in the operator norm.*

PROOF. It suffices to prove our assertion for real functions. Then the operator T and the aforementioned sums are selfadjoint operators. We can assume that $0 \leqslant f(\omega) < 1$. Let us divide the interval $[0,1)$ into equal subintervals of the form $J_{n,k} = [(k-1)/n, k/n)$ and set $\Omega_{n,k} = f^{-1}(J_{n,k})$. For any choice of points $\omega_{n,k} \in \Omega_{n,k}$ for every $a \in H$ with $\|a\| \leqslant 1$ we have

$$\begin{aligned}
&\Bigl|(Ta, a) - \sum_{k=1}^n f(\omega_{n,k})\Pi_{a,a}(\Omega_{n,k})\Bigr| \\
&\qquad = \Bigl|\sum_{k=1}^n \int_{\Omega_{n,k}} f(\omega)\, d\Pi_{a,a}(\omega) - \sum_{k=1}^n f(\omega_{n,k})\Pi_{a,a}(\Omega_{n,k})\Bigr| \\
&\qquad\qquad \leqslant \sum_{k=1}^n \sup_{\omega\in\Omega_{n,k}} |f(\omega) - f(\omega_{n,k})|\Pi_{a,a}(\Omega_{n,k}) \leqslant \frac{1}{n}.
\end{aligned}$$

Since we deal with selfadjoint operators, by Theorem 7.2.6 we obtain the estimate $\bigl\|T - \sum_{k=1}^n f(\omega_{n,k})\Pi(\Omega_{n,k})\bigr\| \leqslant 1/n$. □

Note, however, that the integral with respect to a projection-valued measure is not a Bochner integral with respect to the operator norm.

7.9.4. Proposition. *Let Π be a projection-valued measure on a σ-algebra $\mathcal{B}$ in a space Ω, φ and ψ bounded $\mathcal{B}$-measurable complex functions, and let A and B be the integrals of φ and ψ with respect to the measure Π in the sense defined* *above. Then for all $u, v \in H$ we have*

$$(ABu, v) = \int_\Omega \varphi(\omega)\psi(\omega)\, d\Pi_{u,v}(\omega). \tag{7.9.1}$$

PROOF. It is clear from the previous proposition that $\Pi(S)B = B\Pi(S)$ for all $S \in \mathcal{B}$. It suffices to verify (7.9.1) for functions with finitely many values, which reduces to indicator functions of sets. In the case where $\varphi = I_{S_1}$, $\psi = I_{S_2}$, we first assume that $S_1 \cap S_2 = \varnothing$. Then the right-hand side of (7.9.1) equals zero and the left-hand side is

$$\Pi_{Bu,v}(S_1) = \bigl(\Pi(S_1)Bu, v\bigr) = \bigl(B\Pi(S_1)u, v\bigr) = \bigl(\Pi(S_2)\Pi(S_1)u, v\bigr) = 0.$$

The equality to be proved remains also valid in the case where $S_1 = S_2$. The general case follows from this, since

$$S_1 = M_1 \cup (S_1 \cap S_2),\ S_2 = M_2 \cup (S_1 \cap S_2),$$

where M_1, M_2 and $S_1 \cap S_2$ are pairwise disjoint. □

Let us single out the case where $\Omega = K$ is a compact set on the real line (for example, $[a, b]$) and $\mathcal{B}$ is the Borel σ-algebra of K. In this case we obtain a selfadjoint operator

$$A := \int_K \lambda\, d\Pi(\lambda). \tag{7.9.2}$$

The previous proposition yields that for every $k \in \mathbb{N}$ the operator A^k is written as the integral of λ^k with respect to $d\Pi(\lambda)$. Hence for every polynomial f we obtain

$$f(A) = \int_K f(\lambda)\, d\Pi(\lambda).$$

Since two Borel measures on a compact set with equal integrals of polynomials coincide (see Lemma 3.8.9), we arrive at the following conclusion.

7.9.5. Corollary. *If A is represented in the form* (7.9.2) *with respect to a projection-valued measure Π on K, then such a measure is unique.*

We now show that every selfadjoint operator can be represented in the form of an integral with respect to a projection-valued measure.

7.9.6. Theorem. (THE SPECTRAL DECOMPOSITION OF A SELFADJOINT OPERATOR) *Let A be a selfadjoint operator on a separable Hilbert space $H \neq 0$. Then there exists a unique projection-valued measure Π on $\mathcal{B}(\mathbb{R}^1)$ with $\Pi(\mathbb{R}^1) = I$ vanishing outside some interval such that for every bounded Borel function f we have*

$$f(A) = \int_{\sigma(A)} f(\lambda)\, d\Pi(\lambda) = \int_{\mathbb{R}^1} f(\lambda)\, d\Pi(\lambda). \tag{7.9.3}$$

In particular,

$$A = \int_{\sigma(A)} \lambda\, d\Pi(\lambda) = \int_{\mathbb{R}^1} \lambda\, d\Pi(\lambda). \tag{7.9.4}$$

The measure Π is concentrated on $\sigma(A)$, i.e., $\Pi\bigl(\mathbb{R}^1 \backslash \sigma(A)\bigr) = 0$.

PROOF. Set $\Pi(B) := I_B(A)$. Then $\Pi(B)$ is an orthogonal projection, since $\Pi(B) \geqslant 0$ and $\Pi(B)^2 = \Pi(B)$. If A is realized as the operator of multiplication by a Borel function φ, then $\Pi(B)$ is the operator of multiplication by $I_B \circ \varphi = I_{\varphi^{-1}(B)}$. The measure Π is concentrated on $\sigma(A)$, i.e., $\Pi(B) = 0$ if $B \cap \sigma(A) = \varnothing$, since by Theorem 7.8.6 we can choose φ with values in $\sigma(A)$. If $f = I_M$, where $M \in \mathcal{B}(\mathbb{R}^1)$, then for every $a, b \in H$ we have

$$\int_{\sigma(A)} I_M(t)\, d\Pi_{a,b}(t) = \int_{\mathbb{R}^1} I_M(t)\, d\Pi_{a,b}(t) = \Pi_{a,b}(M) = \bigl(I_M(A)a, b\bigr),$$

i.e., equality (7.9.3) is true for simple functions f. By means of uniform approximations it is easily extended to bounded functions f. The uniqueness of the measure Π follows by the corollary above. $\square$

Obviously, in (7.9.3) or (7.9.4) in place of $\sigma(A)$ we can take any interval $[a, b] \supset \sigma(A)$, for example, the interval $[-\|A\|, \|A\|]$.

For the operator of multiplication by the argument the spectral measure is explicitly calculated in the proof: $\Pi(B)$ is the multiplication by I_B. It is also easy to obtain from the proof explicit expressions for the measures $\Pi_{a,b}$ for the operator of multiplication by φ in $L^2(\mu)$: $\Pi_{a,b} = (a\overline{b} \cdot \mu) \circ \varphi^{-1}$ for all $a, b \in L^2(\mu)$. For example, for the operator of multiplication by the argument the measure $\Pi_{a,b}$ is given by the density $a\overline{b}$ with respect to μ.

If A is an orthogonal projection, then in case $A \neq 0$ and $A \neq I$ we have $\Pi = (I - A)\delta_0 + A\delta_1$, where δ_0 and δ_1 are the Dirac measures at the points 0 and 1. Finally, if A is a selfadjoint operator with an eigenbasis $\{e_n\}$ and eigenvalues $\{\alpha_n\}$, i.e., $Ae_n = \alpha_n e_n$, then $\Pi = \sum_{n=1}^{\infty} P_n \delta_{\alpha_n}$, where P_n is the projection onto the linear span of e_n.

It follows from this theorem and Proposition 7.9.3 that any selfadjoint operator A is the limit in the operator norm of a sequence of finite linear combinations of orthogonal projections. Of course, this can be also seen from the theorem about representation of A as the multiplication by a bounded real function φ: it suffices to uniformly approximate φ by simple functions.

The projection-valued function $\Pi_0(\lambda) := \Pi\bigl((-\infty, \lambda)\bigr)$ is called a *resolution of the identity*. This is a very important characteristic of the operator A. Similarly to the distribution function of a scalar measure, it possesses the following properties:

(i) $\Pi_0(\lambda) \leqslant \Pi_0(\mu)$ if $\lambda \leqslant \mu$,
(ii) $\Pi_0(\lambda_n)x \to \Pi_0(\lambda)x$ for all $x \in H$ if $\lambda_n \uparrow \lambda$.
In addition, $\Pi_0(a) = 0$ if $a < -\|A\|$ and $\Pi_0(b) = I$ if $b > \|A\|$.

Projection-valued measures with equal resolutions of the identity coincide, since scalar measures are uniquely determined by their distribution functions. Moreover, the function Π_0 can be determined by the operator A without recurring to its functional model (but using the functional calculus). To this end, set

$$\Pi_0(\lambda)h = \lim_{n\to\infty} \psi_n(A)h,$$

where ψ_n is a continuous function on the real line defined as follows: $\psi_n(t) = 1$ if $t \leqslant \lambda - 1/n$, $\psi_n(t) = 0$ if $t \geqslant \lambda$, and on $(\lambda - 1/n, \lambda)$ the function ψ_n is linearly interpolated. This shows that bounded Borel functions of A constructed above are uniquely determined by the operator A itself (through the functional calculus) and do not depend on our choice of the functional model (used in our definition of these objects). Independence of the model is also seen from Corollary 7.9.5.

Any function Π_0 with the properties indicated above generates a projection-valued measure similarly to the case where a scalar measure is constructed by its distribution function. Before proving this we observe that for any orthogonal projections P_1 and P_2 in H the condition $P_1 \leqslant P_2$ is equivalent to the inclusion $P_1(H) \subset P_2(H)$. Indeed, if $(P_1 h, h) \leqslant (P_2 h, h)$, then for any h with $P_1 h = h$ we have $\|h\|^2 = (P_1 h, h) \leqslant (P_2 h, h)$, whence $P_2 h = h$, because P_2 is an orthogonal projection. The converse is obvious.

7.9.7. Proposition. *Let* $\Pi_0\colon \mathbb{R}^1 \to \mathcal{P}(H)$ *have properties* (i) *and* (ii) *above and* $\Pi_0(a) = 0$ *and* $\Pi_0(b) = I$ *for some* $a < b$. *Then there is a selfadjoint operator* A *with* $\sigma(A) \subset [a, b]$ *for which* Π_0 *is the resolution of the identity.*

PROOF. The function $\Pi_{x,x}\colon\ \lambda \mapsto \bigl(\Pi_0(\lambda)x, x\bigr)$ for every $x \in H$ is the distribution function of some nonnegative Borel measure μ_x on the real line concentrated on $[a, b]$, moreover, $\|\mu_x\| \leqslant \|x\|^2$. The function $\Pi_{x,y}\colon\ \lambda \mapsto \bigl(\Pi_0(\lambda)x, y\bigr)$ for every $x, y \in H$ is the distribution function of the complex Borel measure $\mu_{x,y}$ on the real line generated by the measure μ_x as follows:

$$4\mu_{x,y} := \mu_{x+y} - \mu_{x-y} + i\mu_{x+iy} - i\mu_{x-iy}.$$

Then $\mu_{x,x} = \mu_x$. For all λ we have

$$\mu_{x,y}\bigl((-\infty, \lambda)\bigr) = \bigl(\Pi_0(\lambda)x, y\bigr).$$

Since measures on the real line with equal distribution functions coincide, we have $\mu_{x+z,y} = \mu_{x,y} + \mu_{z,y}$, $\mu_{\alpha x,y} = \alpha\mu_{x,y}$ and $\mu_{x,y} = \overline{\mu_{y,x}}$. It is clear that $\|\mu_{x,y}\| \leqslant 4$ if $\|x\| \leqslant 1$, $\|y\| \leqslant 1$. Hence for every Borel set B on the real line the function $(x, y) \mapsto \mu_{x,y}(B)$ is linear in x, conjugate-linear in y and continuous in every argument separately. Hence there exists an operator $P(B) \in \mathcal{L}(H)$ such that $\mu_{x,y}(B) = \bigl(P(B)x, y\bigr)$. Since $\mu_{x,x} \geqslant 0$, the operator $P(B)$ is nonnegative selfadjoint. Let us show that $P(B)$ is an orthogonal projection. The operator $P(B) = \Pi_0(\beta) - \Pi_0(\alpha)$ for $B = [\alpha, \beta)$ is an orthogonal projection, which is easily verified with the aid of the equality $\Pi_0(\alpha)\Pi_0(\beta) = \Pi_0(\beta)\Pi_0(\alpha) = \Pi_0(\alpha)$, following from the condition $\Pi_0(\alpha) \leqslant \Pi_0(\beta)$. This yields the equality

$$P(B \cap E) = P(B)P(E) = P(E)P(B)$$

for semi-intervals. Let $x, y \in H$. Let us consider two complex Borel measures $E \mapsto \bigl(P(B \cap E)x, P(E)y\bigr)$ and $E \mapsto \bigl(P(B)x, P(E)y\bigr)$. If B is a semi-interval $[\alpha, \beta)$, then these two measures have equal values on semi-intervals, hence coincide, i.e., $P(B \cap E) = P(B)P(E) = P(E)P(B)$ for all Borel sets E. Repeating this reasoning for a fixed Borel set E, we conclude that the equality remains valid for all Borel sets B. In particular, $P(B) = P(B)^2$. It is clear that $\Pi_0(\lambda) = P\bigl((-\infty, \lambda)\bigr)$. Now set

$$A := \int_{[a,b]} \lambda \, dP(\lambda).$$

By Corollary 7.9.5 the operator A generates the projection-valued measure P and has the resolution of the identity Π_0. □

7.10. Complements and Exercises

7.10(i). The structure of the spectrum

Let us discuss the structure of the spectrum of a bounded operator A on an infinite-dimensional separable Hilbert space H. The set $\sigma(A)$ is nonempty and compact in $\mathbb{C}$. On the other hand, every nonempty compact set $K \subset \mathbb{C}$ is the spectrum of some operator $A \in \mathcal{L}(H)$, since we can find a finite or countable everywhere dense set of points λ_n in K, take an orthonormal basis $\{e_n\}$ in H and define a bounded diagonal operator on H by the formula $Ae_n = \lambda_n e_n$. Its spectrum is the closure of $\{\lambda_n\}$ (Exercise 7.10.57), which is exactly K. It has been proved by Gowers and Maurey, there exist infinite-dimensional separable Banach spaces in which the spectrum of every bounded operator is finite or countable.

If the set $(A-\lambda I)(H)$ is dense and the operator $(A-\lambda I)^{-1}$ is continuous on this set, then it extends to a bounded operator, which will serve as the inverse to $A-\lambda I$. An important part of the spectrum of A is the *point spectrum*, i.e., the set $\sigma_p(A)$ of eigenvalues (which, as we know, can be absent in the infinite-dimensional case). The remaining points $\lambda \in \sigma(A)$ belong to the spectrum because the mapping $(A-\lambda I)^{-1}\colon\ (A-\lambda I)(H) \to H$ is either discontinuous or defined on a set that is not dense (although is continuous). The *continuous spectrum* $\sigma_c(A)$ is usually defined as the set of all numbers $\lambda \in \sigma(A)\backslash\sigma_p(A)$ for which $A-\lambda I$ has a dense range, but the inverse operator is discontinuous on it. Then the *residual spectrum* $\sigma_r(A)$ is $\sigma(A)\backslash\bigl(\sigma_p(A)\cup\sigma_c(A)\bigr)$, i.e., the collection of numbers λ for which the range of the operator $A-\lambda I$ is not dense (in this case $(A-\lambda I)^{-1}$ can be continuous on this range or discontinuous). However, one encounters different partitions of the spectrum in the literature. For example, sometimes the residual spectrum is defined to consist of those λ for which the range of $A-\lambda I$ is not dense, but the inverse operator is bounded.

Let us describe the structure of the set of eigenvalues.

7.10.1. Theorem. *Let A be a bounded operator on an infinite-dimensional separable Hilbert space H. Then the set $\sigma_p(A)$ of all eigenvalues of A is a countable union of compact sets.*

Conversely, every bounded set that is a countable union of compact sets serves as the set of all eigenvalues of some bounded operator on H.

PROOF. The set of eigenvalues is bounded as a subset of the spectrum. The closed unit ball U in H with the weak topology is a metrizable compact space. Hence its open subset $U\backslash\{0\}$ can be represented as a countable union of closed (in the weak topology) parts $K_n \subset U$. The sets K_n are compact in the weak topology.

A point $\lambda \in \mathbb{C}$ is an eigenvalue of A precisely when $\|Ax - \lambda x\| = 0$ for some $x \in U\backslash\{0\}$. Hence $\sigma_p(A)$ is the union of the projections to $\mathbb{C}$ of the sets

$$M_n := \bigl\{(x, \lambda) \in K_n \times D\colon\ \|Ax - \lambda x\| = 0\bigr\},$$

where D is the closed disc in $\mathbb{C}$ with the center at the origin and radius $\|A\|$. We observe that M_n is compact if K_n is equipped with the weak topology. This is seen from the fact that $K_n \times D$ is compact when we equip K_n with the weak topology and M_n is closed in this product, since it is specified by the conditions $\lambda(x, e_i) = (x, A^*e_i)$, where $\{e_i\}$ is an orthonormal basis in H. Therefore, the projection of M_n to $\mathbb{C}$ is also compact.

We now show that every bounded set P equal to the union of compact sets $S_n \subset \mathbb{C}$ coincides with the point spectrum of some bounded operator on a separable Hilbert space. It suffices to show this for every S_n separately, since the direct sum of uniformly bounded operators A_n in Hilbert spaces H_n has for the family of eigenvalues the union of the point spectra of A_n. Thus, we shall deal with a single compact set S. We can assume that it is not empty and is contained in $D := \{z\colon\ |z| < 1\}$. Let us consider the Bergman space $A^2(D)$ of all functions $f \in L^2(D)$ holomorphic in D, where D is equipped with Lebesgue measure (Example 5.2.2). We know that $A^2(D)$ is a separable Hilbert space with the norm from $L^2(D)$, i.e.,

$$\|f\|^2_{A^2(D)} := \int_D |f(x + iy)|^2\, dx\, dy.$$

Let us consider the operator T on the dual to the space $A^2(D)$ (so that T acts on distributions, not on functions) given by the formula $T\psi(f) := \psi(zf)$, where $(zf)(z) = zf(z)$. Now we do not identify $A^2(D)$ with its dual. Actually, T is the operator adjoint to the multiplication by the argument on $A^2(D)$. However, the operator T has too many eigenvalues: every point λ in D turns out to be an eigenvalue, since the functional $\psi_\lambda\colon\ f \mapsto f(\lambda)$ satisfies the equality $T\psi_\lambda = \lambda\psi_\lambda$. The continuity of this functional is clear from estimate (5.2.1) in Example 5.2.2. Hence it is natural to take for H the closed linear subspace in $A^2(D)^*$ generated by the functionals ψ_λ with $\lambda \in S$. It is clear that $T(H) \subset H$ and all elements of S remain eigenvalues of $T|_H$. Let us verify that there are no other eigenvalues. It suffices to make sure that the operator T on H has no eigenvalues on the unit circumference and for every $\lambda \in D$ the kernel of $T - \lambda I$ in the whole $A^2(D)^*$ is one-dimensional. Since T is the adjoint to the operator T_1 of multiplication by the argument on $A^2(D)$, we have to verify that the range of $T_1 - \lambda I$ is dense if $|\lambda| = 1$ and has a one-dimensional orthogonal complement if $|\lambda| < 1$. Suppose that there exists λ with $|\lambda| = 1$ and a unit vector $g \in A^2(D)$ such that $(T_1 f - \lambda f, g) = 0$ for all $f \in A^2(D)$, i.e.,

$$\int_D (x + iy) f(x + iy)\overline{g(x + iy)}\, dx\, dy = \lambda \int_D f(x + iy)\overline{g(x + iy)}\, dx\, dy$$

for all $f \in A^2(D)$. In particular, for $f = g$ we obtain in the right-hand side a number the absolute value of which equals 1. The left-hand side is strictly less, since $|x + iy| < 1$ if $x + iy \in D$. Let now $|\lambda| < 1$. Since the functional $f \mapsto f(\lambda)$ is continuous, its kernel H_λ is closed and has codimension 1. In addition, every function in H_λ belongs to the range of the operator $T_1 - \lambda I$. Indeed, for every function $f \in H_\lambda$ the function $g(z) = (z-\lambda)^{-1} f(z)$ belongs to $A^2(D)$ (it is holomorphic and square integrable in D, since $f(\lambda) = 0$) and satisfies the equality $f = (T_1 - \lambda I)g$. $\square$

Close considerations give the following result (see [**708**]).

7.10.2. Theorem. (i) *For every operator $A \in \mathcal{L}(l^2)$, the set $\sigma_c(A)$ is a countable intersection of open sets.*

(ii) *Let $K \subset \mathbb{C}$ be a nonempty compact set such that $K = P \cup C \cup R$, where P, C and R are pairwise disjoint, P is a countable union of compact sets, and C is a countable intersection of open sets. Then there exists an operator $A \in \mathcal{L}(l^2)$ such that $\sigma_p(A) = P$, $\sigma_c(A) = C$, and $\sigma_r(A) = R$.*

7.10(ii). Commuting selfadjoint operators

The class of selfadjoint operators is contained in the class of *normal* operators: bounded operators B on a Hilbert space such that $BB^* = B^*B$. The latter class contains also all unitary operators. It turns out that normal operators are also unitarily isomorphic to operators of multiplication. This fact is derived below from a more general assertion about simultaneous representation of commuting selfadjoint operators in the form of multiplication by a function. We first prove an auxiliary result on projection-valued measures. To projection-valued measures one can extend some (but not all!) results of the usual measure theory.

7.10.3. Proposition. *Suppose that on an algebra $\mathcal{R}$ in a space Ω we are given an additive set function Π with values in the set of orthogonal projections in a Hilbert space H. Suppose that for every $a, b \in H$ the complex function*

$$R \mapsto \Pi_{a,b}(R) := \bigl(\Pi(R)a, b\bigr)$$

is countably additive on $\mathcal{R}$. Then the function Π has a unique extension to a projection-valued measure on the σ-algebra $\sigma(\mathcal{R})$ generated by $\mathcal{R}$.

PROOF. For any fixed $a, b \in H$, the function $\Pi_{a,b}$ has a unique extension to a countably additive complex measure on $\sigma(\mathcal{R})$ denoted also by $\Pi_{a,b}$. Indeed, if $a = b$, then the function $\Pi_{a,a}$ on $\mathcal{R}$ is countably additive, nonnegative and bounded, since $\bigl(\Pi(R)a, a\bigr) \leqslant \bigl(\Pi(\Omega)a, a\bigr)$. By Theorem 2.4.6 it uniquely extends to a bounded measure on $\sigma(\mathcal{R})$. The formulas

$$2\mathrm{Re}\,\Pi_{a,b} = \Pi_{a+b,a+b} - \Pi_{a,a} - \Pi_{b,b}, \quad 2\mathrm{Im}\,\Pi_{a,b} = \Pi_{a+ib,a+ib} - \Pi_{a,a} + \Pi_{b,b}$$

give extensions of $\Pi_{a,b}$ to $\sigma(\mathcal{R})$. Note that $\Pi_{a,b} = \overline{\Pi_{b,a}}$ on $\sigma(\mathcal{R})$, since this is true on $\mathcal{R}$. Therefore, for every $S \in \sigma(\mathcal{R})$ there exists a bounded selfadjoint operator $\Pi(S)$ with $\bigl(\Pi(S)a, b\bigr) = \Pi_{a,b}(S)$ (see Lemma 7.2.2). It follows from our construction that $0 \leqslant \Pi(S) \leqslant I$. Let us show that $\Pi(S)$ is an orthogonal projection. Denote by $\mathcal{M}$ the class of sets $S \in \sigma(\mathcal{R})$ with this property. Then $\mathcal{M}$ contains the algebra $\mathcal{R}$. In addition, $\mathcal{M}$ is a monotone class: if sets M_n in $\mathcal{M}$ either increase to M or decrease to M, then $M \in \mathcal{M}$. Indeed, in the first case $\Pi(M_{n+1}) = \Pi(M_n) + \Pi(M_{n+1} \backslash M_n) \geqslant \Pi(M_n)$, i.e., $\Pi(M_n)$ are projections onto increasing closed subspaces H_n and hence M is the projection onto the closure of the union of H_n. In the second case the reasoning is similar. $\square$

7.10.4. Corollary. *Let Π' and Π'' be projection-valued measures on σ-algebras $\mathcal{A}'$ and $\mathcal{A}''$ in spaces Ω' and Ω''. Suppose that $\Pi'(S')$ and $\Pi''(S'')$ commute for all $S' \in \mathcal{A}'$ and $S'' \in \mathcal{A}''$. Then on the σ-algebra $\mathcal{A} = \mathcal{A}' \otimes \mathcal{A}''$ in the space $\Omega = \Omega' \times \Omega''$ there is a projection-valued measure Π such that $\Pi(S' \times S'') = \Pi'(S')\Pi''(S'')$ for all $S' \in \mathcal{A}'$ and $S'' \in \mathcal{A}''$.*

PROOF. Products of the form $S' \times S''$, where $S' \in \mathcal{A}'$, $S'' \in \mathcal{A}''$, constitute a semi-algebra $\mathcal{R}_0$, on which Π can be defined by $\Pi(S' \times S'') = \Pi'(S')\Pi''(S'')$. It is readily seen that we have obtained an additive projection-valued function. The sets in the algebra $\mathcal{R}$ generated by $\mathcal{R}_0$ have the form $R = R_1 \cup \cdots \cup R_n$, where $R_i \in \mathcal{R}_0$ are disjoint. Let us extend Π to $\mathcal{R}$ by the natural formula $\Pi(R) = \Pi(R_1) + \cdots + \Pi(R_n)$. We observe that $\Pi(R_i)\Pi(R_j) = 0$ if $i \neq j$. Hence $\Pi(R)$ is an orthogonal projection. Proposition 2.3.7 yields that the nonnegative scalar measures $R \mapsto \big(\Pi(R)a, a\big)$ are countably additive on $\mathcal{R}$ for all $a \in H$. This gives countably additivity complex measures $R \mapsto \big(\Pi(R)a, b\big)$, $a, b \in H$. $\square$

7.10.5. Lemma. *Let A be a selfadjoint operator with a cyclic vector and let B be a selfadjoint operator commuting with A. Then B is a Borel function of A.*

PROOF. We can assume that A is the operator of multiplication by the argument on $L^2(\mu)$ for some measure μ on an interval. Set $\psi = B(1)$, where we choose a Borel version of the function $B(1) \in L^2(\mu)$. We show that B coincides with the operator of multiplication by ψ. First we verify that for every function $p_k\colon\ t \mapsto t^k$ one has the equality $Bp_k = \psi p_k$. For $k = 0$ this is true. If the equality is true for some $k \geqslant 0$, then it remains valid for $k+1$, because we have $Bp_{k+1} = BAp_k = ABp_k = A(\psi p_k) = \psi p_{k+1}$. Thus, $Bp = \psi \cdot p$ for every polynomial p. This yields that $Bf = \psi \cdot f$ for every function $f \in L^2(\mu)$. Indeed, there exists a sequence of polynomials f_k converging to f in $L^2(\mu)$, which gives convergence of Bf_k to Bf in $L^2(\mu)$ and convergence of $\psi \cdot f_k$ to $\psi \cdot f$ in $L^1(\mu)$. From the equality $Bf_k = \psi \cdot f_k$ we obtain the equality $Bf = \psi \cdot f$ a.e. Since this is true for all $f \in L^2(\mu)$, one has $\psi \in L^\infty(\mu)$. This can be easily verified without using the property that the operator B is bounded, but this property gives at once the estimate $|\psi(t)| \leqslant \|B\|$ for μ-a.e. t, since if the set $M = \{t\colon\ |\psi(t)| > \|B\|\}$ has positive measure, then $\|B\| \cdot \|I_M\|_{L^2} < \|\psi I_M\|_{L^2} = \|BI_M\|_{L^2} \leqslant \|B\| \cdot \|I_M\|_{L^2}$. $\square$

7.10.6. Lemma. *Suppose that selfadjoint operators A and B on a separable Hilbert space commute. Then for any bounded Borel functions φ and ψ the operators $\varphi(A)$ and $\psi(B)$ commute.*

PROOF. According to Exercise 7.10.78 there are polynomials p_n such that the operators $p_n(A)$ converge to $\varphi(A)$ on every vector. Then

$$B\varphi(A)x = \lim_{n\to\infty} Bp_n(A)x = \lim_{n\to\infty} p_n(A)Bx = \varphi(A)Bx.$$

Thus, B commutes with $\varphi(A)$. Applying the proved assertion once again, we obtain the equality $\psi(B)\varphi(A) = \varphi(A)\psi(B)$. $\square$

7.10.7. Theorem. *Suppose that selfadjoint operators $A_1, \ldots, A_n$ on a separable Hilbert space $H \neq 0$ commute. Then there exists a bounded nonnegative Borel measure μ on $\mathbb{R}^n$ along with a unitary isomorphism $J\colon H \to L^2(\mu)$ and bounded Borel functions $\varphi_1, \ldots, \varphi_n$ on $\mathbb{R}^n$ such that the operators JA_iJ^{-1} are the operators of multiplication by φ_i for all $i = 1, \ldots, n$.*

PROOF. Suppose first that there is a unit vector h such that the set of finite linear combinations of all vectors of the form $A_1^{k_1} \cdots A_n^{k_n} h$, $k_i = 0, 1, \ldots$, is dense in H. Let us write A_i in the form of integrals with respect to projection-valued measures Π_i on the real line. On $\mathbb{R}^n$ we can define a projection-valued measure Π with the aid of Corollary 7.10.4 and induction. We observe that

$$A_i = \int_{\mathbb{R}^1} t_i \, d\Pi_i(t) = \int_{\mathbb{R}^n} t_i \, d\Pi(t), \quad i = 1, \ldots, n.$$

Let $\mu = \Pi_{h,h}$. The measure μ is concentrated on the set $\prod_{i=1}^n [-\|A_i\|, \|A_i\|]$. Let us define a mapping $J\colon H \to L^2(\mu)$ by the formula

$$J(A_1^{k_1} \cdots A_n^{k_n} h) := p_{k_1,\ldots,k_n}, \quad \text{where} \quad p_{k_1,\ldots,k_n}(t_1, \ldots, t_n) = t_1^{k_1} \cdots t_n^{k_n},$$

then we extend it by linearity to the linear span of such vectors. We observe that for any finite linear combination of the indicated vectors we have the equality

$$\Bigl|\sum c_{k_1,\ldots,k_n} A_1^{k_1} \cdots A_n^{k_n} h\Bigr|^2 = \int \Bigl|\sum c_{k_1,\ldots,k_n} p_{k_1,\ldots,k_n}\Bigr|^2 \, d\mu,$$

since by relation (7.9.1) we have

$$\begin{aligned}(A_1^{k_1} \cdots A_n^{k_n} h, A_1^{l_1} \cdots A_n^{l_n} h) &= (A_1^{k_1+l_1} \cdots A_n^{k_n+l_n} h, h) \\ &= \int_{\mathbb{R}^n} t_1^{k_1+l_1} \cdots t_n^{k_n+l_n} \, \mu(dt).\end{aligned}$$

The obtained equality means that the mapping J is well-defined and preserves the inner product. The range of J is everywhere dense in $L^2(\mu)$, since it contains all polynomials. Hence J extends to a unitary isomorphism. In the general case we can decompose H into the orthogonal sum of closed subspaces invariant with respect to all operators A_i and possessing the aforementioned property. Similarly to the case of a single operator, this is done with the aid of Zorn's lemma. □

7.10.8. Corollary. *Every normal operator S on a separable Hilbert space $H \neq 0$ is unitarily equivalent to the operator of multiplication by a bounded complex Borel function on the space $L^2(\mu)$ for some bounded nonnegative Borel measure μ on $\mathbb{C}$.*

PROOF. The operators $A = S + S^*$ and $B = i^{-1}(S - S^*)$ are selfadjoint and commute. In addition, $S = A/2 + iB/2$. It remains to represent simultaneously A and B in the form of multiplication. □

7.10.9. Corollary. *Every unitary operator U on a separable Hilbert space $H \neq 0$ is unitarily equivalent to the operator of multiplication by a Borel function ζ on the space $L^2(\mu)$ for some bounded Borel measure μ on $\mathbb{C}$ or on $\mathbb{R}^1$ such that*

$|\zeta(t)| = 1$ for μ-a.e. t. In addition, there is a representation $U = \exp(iB)$, where B is a selfadjoint operator.

PROOF. By the previous corollary we can assume that U is the multiplication by a bounded function ζ on $L^2(\mu)$ for some Borel measure μ on $\mathbb{C}$. Then $|\zeta(t)| = 1$ μ-a.e. Hence ζ can be written in the form $\zeta = \exp(ig)$, where g is a Borel function with values in $[0, 2\pi]$. For B we take the operator of multiplication by g. Now we write B in the form of multiplication by a function on an interval. $\square$

By using the projection-valued measure Π of the selfadjoint operator B from this corollary, we write the operator U in the form

$$U = \int_{\sigma(B)} \exp(i\lambda)\, d\Pi(\lambda).$$

We now extend the previous theorem to infinite families of commuting operators.

7.10.10. Theorem. *Suppose we are given a countable set of commuting selfadjoint operators A_n on a separable Hilbert space $H \neq 0$. Then there exists a bounded nonnegative Borel measure μ on $[0,1]^\infty$ along with a unitary isomorphism $J\colon H \to L^2(\mu)$ and bounded Borel functions φ_n on $[0,1]^\infty$ such that the operators JA_nJ^{-1} are the operators of multiplication by φ_n for all n.*

PROOF. Without loss of generality we can assume that $\sigma(A_n) \subset [0,1]$. As in the previous theorem, on $[0,1]^\infty$ we construct a Borel projection-valued measure Π with respect to which the integrals of the coordinate functions give our operators A_n. The only difference is that for the initial algebra on which the measure Π is defined is constituted by the unions of finite powers of $\mathcal{B}([0,1])$ (the measure Π is defined on them by the previous theorem). As in the case of a finite collection of operators, the general case reduces to the situation where there is a unit vector h with the property that the linear span of all possible vectors of the form $A_1^{k_1} \cdots A_n^{k_n} h$ is dense in H. In this situation we employ the same isomorphism J as in the proof for a finite collection. Its range consists of polynomials in finitely many variables, which is everywhere dense in $L^2(\mu)$. Hence the proof is completed as above. $\square$

Let us consider arbitrary collections of operators.

7.10.11. Theorem. *Suppose we are given a collection $\mathcal{T}$ of commuting selfadjoint operators on a separable Hilbert space H. Then there exists a selfadjoint operator A on H and, for every $T \in \mathcal{T}$, there is a bounded Borel function φ_T on the real line such that $T = \varphi_T(A)$ for all $T \in \mathcal{T}$.*

PROOF. We first consider the case of a countable collection of commuting selfadjoint operators A_n. As shown above, we can assume that the operators A_n are the operators of multiplication by bounded Borel functions φ_n on $L^2(\mu)$, where μ is a Borel measure on $[0,1]^\infty$. We now use the following fact (see [**73**, Corollary 6.8.8]): there exists a Borel isomorphism G between $[0,1]^\infty$ and the interval $[0,1]$. Let ν be the image of the measure μ under this isomorphism. Then the operators A_n become the operators of multiplication by the Borel functions $\psi_n = \varphi_n \circ G^{-1}$ in $L^2(\nu)$ with the measure ν on the interval, hence they become functions of the operator of multiplication by the argument on $L^2(\nu)$.

In the general case we find in $\mathcal{T}$ a countable family of operators $\mathcal{T}_0$ with the following property: for every $T \in \mathcal{T}$, there exists a sequence of operators $A_n \in \mathcal{T}_0$ converging to T in the weak operator topology, i.e., $(A_n x, y) \to (Tx, y)$ for all $x, y \in H$. This is possible, since, as is readily seen, the weak operator topology is metrizable on balls. As shown above, we can assume that all operators in $\mathcal{T}_0$ have the form of multiplication by some functions $\psi \circ \varphi$, where ψ is a bounded Borel function and φ is the Borel function on the interval determining the operator on $L^2(\nu)$ functions of which are the operators in $\mathcal{T}_0$. We can choose a Borel version of φ such that the norm of every operator in $\mathcal{T}_0$ will equal $\sup_x |\psi \circ \varphi(x)|$ for the corresponding function ψ. Let $T \in \mathcal{T}$. Let us find a sequence $\{T_n\} \subset \mathcal{T}_0$ converging to T in the weak operator topology. Then $\{\psi_n \circ \varphi\}$ converges weakly in $L^2(\nu)$ to some limit g. The sequence $\{T_n\}$ is bounded in the operator norm, which gives the boundedness of $\{\psi_n \circ \varphi\}$ in $L^\infty(\nu)$. Then $g \in L^\infty(\nu)$, since the sequence $S_n \circ \varphi$ of the arithmetic means of some subsequence in $\{\psi_n \circ \varphi\}$ converges in the norm of $L^2(\nu)$ (see Example 6.10.33). It is clear that the operator T is given by the multiplication by g, but we have to verify that $g = \psi \circ \varphi$ for some bounded Borel function ψ. Passing to a subsequence, we can assume that $S_n \circ \varphi \to g$ ν-a.e. By Luzin's theorem, the set of convergence contains compact sets K_j with $\nu(K_j) \to \nu([0,1])$ on which the function φ is continuous. The sets $\varphi(K_j)$ are compact, $E = \bigcup_{j=1}^\infty \varphi(K_j)$ is a Borel set, and for every $y \in E$ the sequence $\{S_n(y)\}$ converges. Let us denote the limit by $\psi(y)$. Outside E we define ψ by zero. We have obtained a bounded Borel function. For any $x \in \bigcup_{j=1}^\infty K_j$ we have $\psi\bigl(\varphi(x)\bigr) = \lim\limits_{n\to\infty} \psi_n\bigl(\varphi(x)\bigr) = g(x)$, i.e., $\psi\bigl(\varphi(x)\bigr) = g(x)$ for ν-a.e. x. □

7.10.12. Corollary. *Every normal operator T on a separable Hilbert space H has the form $T = f(A)$, where A is a selfadjoint operator on H and f is a bounded complex Borel function.*

PROOF. Since $T = A_1 + iA_2$, where A_1 and A_2 are commuting selfadjoint operators, we can apply the theorem above (of course, here we need its simplest case). □

Hence T can be written in the form of an integral (7.9.3) of f with respect to a projection-valued measure Π. Consequently, T can be written as the integral

$$T = \int_{\sigma(T)} z \, dP(z)$$

with respect to the projection-valued measure P on $\sigma(T)$ defined as follows: $P(B) := \Pi\bigl(f^{-1}(B)\bigr)$.

7.10(iii). Operator ranges in a Hilbert space

Let us apply the polar decomposition to show that the range of every bounded operator on a Hilbert space coincides with the range of some selfadjoint operator.

7.10.13. Lemma. *For every bounded operator A on a Hilbert space H we have*

$$A(H) = |A^*|(H).$$

PROOF. We have $A^* = V|A^*|$, where the operator V isometrically maps the subspace $E_1 := (\operatorname{Ker} A^*)^\perp$ onto $E_2 := \overline{A^*(H)}$ and equals zero on $\operatorname{Ker} A^*$. Then $A = |A^*|V^*$, whence $A(H) \subset |A^*|(H)$. On the other hand, we have $\operatorname{Ker}|A^*| = \operatorname{Ker} A^*$, hence $|A^*|(H) = |A^*|(E_1)$. Since V^* isometrically maps E_2 onto E_1, we have $|A^*|(H) = |A^*|V^*(E_2) = A(E_2) \subset A(H)$. □

In accordance with a general result for Banach spaces from §6.10(i), a linear subspace L of a Hilbert space H is the range of a bounded operator on H precisely when it is a Hilbert space continuously embedded into H. Indeed, the quotient space of H with respect to the closed subspace H_0 is also Hilbert (it can be identified with $H_0^\perp$). In addition, if a Hilbert space E is continuously embedded into H, then there is an operator $A \in \mathcal{L}(H)$ with $A(H) = E$ (Exercise 6.10.164). Our next result gives a more constructive condition.

7.10.14. Proposition. *A linear subspace L in a Hilbert space H is the range of a bounded operator on H precisely when there exists a sequence of pairwise orthogonal closed subspaces $H_n \subset H$ such that*

$$L = \Bigl\{\sum_{n=1}^\infty x_n\colon\ x_n \in H_n, \sum_{n=1}^\infty 4^n\|x_n\|^2 < \infty\Bigr\}.$$

PROOF. If L has the indicated form, then $L = A(H)$, where $A = \sum_{n=1}^\infty 2^{-n}P_n$ and P_n is the projection onto H_n.

Let L be the range of a bounded operator. As noted above, we can assume that $L = A(H)$, where A is a nonnegative selfadjoint operator. The general case easily reduces to the case of separable H and an operator A with a cyclic vector. Hence we can assume that A is the operator of multiplication by the argument on the space $L^2(\mu)$ for some bounded Borel measure on $[0,1]$. Let us take for H_n the subspace in $L^2(\mu)$ consisting of the functions vanishing outside $(2^{-n}, 2^{1-n}]$. Then $Ax = \sum_{n=1}^\infty P_nAx$, $P_nAx \in H_n$, and $\|P_nAx\| \leqslant 2^{1-n}\|P_nx\|$. Hence $4^n\|P_nAx\|^2 \leqslant 4\|P_nx\|^2$, which gives a convergent series. On the other hand, if $y = \sum_{n=1}^\infty y_n$, where $y_n \in H_n$ and $\sum_{n=1}^\infty 4^n\|y_n\|^2 < \infty$, then $y_n = Ax_n$, where $x_n \in H_n$ and $\|x_n\| \leqslant 2^n\|y_n\|$, since $x_n(t) = t^{-1}y_n(t)$. Therefore, we have the bound $\sum_{n=1}^\infty \|x_n\|^2 \leqslant \sum_{n=1}^\infty 4^n\|y_n\|^2 < \infty$. Hence the series of the pairwise orthogonal vectors x_n converges to some vector x and $Ax = y$. □

In case of a Hilbert space the factorization Theorem 6.10.5 can be sharpened in the following way (this result was obtained in [**686**]).

7.10.15. Theorem. *Let A and B be two bounded operators on a Hilbert space H. The following conditions are equivalent*:

(i) *$A(H) \subset B(H)$,*

(ii) *$A = BC$ for some $C \in \mathcal{L}(H)$,*

(iii) *there exists a number* $\lambda \geqslant 0$ *such that*

$$(A^*x, A^*x) \leqslant \lambda^2 (B^*x, B^*x) \quad \textit{for all } x \in H.$$

PROOF. The equivalence of (i) and (ii) is seen from Theorem 6.10.5, since in case of a Hilbert space it gives a continuous linear operator $C = B_0^{-1}A$ from H onto $(\operatorname{Ker} B)^\perp$, where B_0 is the restriction of B to $(\operatorname{Ker} B)^\perp$ and B_0^{-1} is the algebraic inverse. Next, (ii) yields (iii) at once. Let (iii) be fulfilled. On the linear space $B^*(H)$ we have the linear mapping $D\colon\ B^*x \mapsto A^*x$. This mapping is well-defined, because $A^*x = 0$ if $B^*x = 0$. In addition, $\|Dy\| \leqslant \lambda\|y\|$ if $y \in B^*(H)$, hence D can be extended to a bounded operator on the closure of $B^*(H)$. Next we extend D to a bounded operator on all of H, defining D by zero on $B^*(H)^\perp = \operatorname{Ker} B$. Then $DB^* = A^*$, whence $A = BD^*$. In part of this reasoning we could refer to results from §6.10(i). □

Note that in (iii) it is important to have an estimate for the adjoint operator, but not for the original one (to see this, it suffices to take $A = I$ and an isometry B with $B(H) \neq H$). From the previous theorem and Corollary 7.5.4 we obtain the following result efficient in estimating the rate of decreasing of eigenvalues of compact selfadjoint operators.

7.10.16. Corollary. *Suppose that* A *and* B *are compact operators on a Hilbert space* H *such that* $A(H) \subset B(H)$. *Let* $\alpha_1^+ \geqslant \alpha_2^+ \geqslant \cdots > 0$ *and* $\beta_1^+ \geqslant \beta_2^+ \geqslant \cdots > 0$ *be positive eigenvalues of the operators* $|A|$ *and* $|B|$, *respectively, written in the decreasing order taking into account their multiplicities. Then there exists* $C > 0$ *such that* $\alpha_n^+ \leqslant C\beta_n^+$.

PROOF. The theorem gives the estimate $AA^* \leqslant \lambda^2 BB^*$. Now it is important that the operators AA^* and A^*A have the same nonzero eigenvalues (Exercise 7.10.65), i.e., $|A^*|$ and $|A|$ have common nonzero eigenvalues; the same is true for the pair of operators BB^* and B^*B. According to Corollary 7.5.4, the indicated estimate yields the inequalities $\lambda_n(AA^*) \leqslant \lambda^2\lambda_n(BB^*)$ for positive eigenvalues of the operators AA^* and BB^* written in the order of decreasing. □

7.10.17. Example. Let $W_{2\pi}^{2,1}[0, 2\pi]$ be the class of all absolutely continuous complex functions f on $[0, 2\pi]$ with $f' \in L^2[0, 1]$ and $f(0) = f(2\pi)$. If the range of a bounded operator A on $L^2[0, 2\pi]$ belongs to $W_{2\pi}^{2,1}[0, 2\pi]$, then A is a compact operator and for the positive eigenvalues α_n^+ of the operator $|A|$ written in the order of decreasing taking into account their multiplicities, for some $C > 0$ one has the estimate $\alpha_n^+ \leqslant Cn^{-1}$.

For this we observe that $W_{2\pi}^{2,1}[0, 2\pi]$ is the range of the operator B with eigenfunctions $\exp(ikt)$ and eigenvalues $\beta_k = k^{-1}$, $k \neq 0$, $\beta_0 = 1$ (Exercise 9.10.32).

Let us now prove the following theorem due to von Neumann [**701**].

7.10.18. Theorem. *Suppose that a bounded operator* A *on a separable Hilbert space has a non-closed range. Then there exists a unitary operator* U *such that* $A(H) \cap UA(H) = 0$.

PROOF. We start with a simple explicit example of such an operator with a dense range in $L^2[-\pi,\pi]$. Let $e_n(t) = e^{int}$ and let A be given by $Ae_n = e^{-n^2}e_n$. It is readily seen that all functions in $A(H)$ are real-analytic on $[-\pi,\pi]$. Let U be the operator of multiplication by $\operatorname{sign} t$. Then the range of A and UA intersect only by zero. This example can be easily modified to make the range of A not only dense, but also containing an infinite-dimensional closed subspace. For this it suffices take the direct sum of countably many copies of A.

The main step of the proof consists in verification of the following interesting fact: if the range of A is dense, non-closed and contains an infinite-dimensional closed subspace, then for every bounded operator B with a non-closed range there exists a unitary operator W such that $WB(H) \subset A(H)$. For this we apply Proposition 7.10.14 to $L = B(H)$ and take the corresponding pairwise orthogonal closed subspaces H_n. Since $B(H)$ is not closed, there are infinitely many nonzero subspaces H_n. Let $\mathbb{N} = \bigcup_{i=1}^{\infty} \Omega_i$ be a partition into countably many countable parts and let S_n be the set Ω_i which contains n. Set $H'_n := \bigoplus_{i\in S_n} H_i$. Applying the cited proposition to the pairwise orthogonal subspaces H'_n, we obtain an operator B' with $B(H) \subset B'(H)$, but now all subspaces H'_n are infinite-dimensional.

We now apply the same proposition to $A(H)$ and take the corresponding nonzero pairwise orthogonal closed subspaces E_n. We observe that among E_n there is at least one infinite-dimensional. Indeed, if all E_n were finite-dimensional, we would obtain a compact operator $C := \sum_{n=1}^{\infty} 2^{-n} P_{E_n}$ with $C(H) = A(H)$, as shown in the cited proposition. However, the range of a compact operator cannot contain an infinite-dimensional closed subspace, because it is covered by countably many compact sets. We can assume that E_1 is infinite-dimensional. Then we can take pairwise orthogonal infinite-dimensional closed subspaces $L_i \subset E_1$ such that $E_1 = \bigoplus_{i=1}^{\infty} L_i$. Set $E'_i := E_i \bigoplus L_i$ if $i > 2$ and $E'_1 := L_1$. We have obtained pairwise orthogonal infinite-dimensional closed subspaces. It is easy to see that for the corresponding operator A' in the cited proposition we have $A'(H) \subset A(H)$. A unitary operator W can be defined by means of unitary isomorphisms between H'_i and E'_i. Then $B(H) \subset B'(H) \subset A'(H) \subset A(H)$. For completing the proof it remains to transform $A(H)$ by the unitary operator W to $A_0(H)$, where A_0 is an operator with a dense range such that $A_0(H) \cap U_0A_0(H) = 0$ for some unitary operator U_0. For U we take $W^{-1}U_0W$. □

This result of von Neumann was further developed by Dixmier [**682**], [**683**] (see also Exercise 7.10.105). The proof above follows Fillmore, Williams [**687**].

Let us observe that in real spaces the same fact is true with orthogonal operators in place of unitary operators. To see this, we can write a real space as the sum of two infinite-dimensional closed subspaces and introduce the corresponding complex structure.

7.10(iv). Hilbert–Schmidt operators and nuclear operators

In this subsection we discuss two classes of compact operators on Hilbert spaces, both important for applications and interesting. One of several equivalent definitions of these classes is connected with the behavior of eigenvalues.

7.10.19. Definition. *Let A be a compact operator on a Hilbert space H (real or complex) and let $\{s_n(A)\}$ be all eigenvalues of the operator $|A|$. We shall call A a Hilbert–Schmidt operator if*

$$\sum_{n=1}^{\infty} s_n(A)^2 < \infty.$$

The operator A will be called a nuclear or trace class operator if

$$\sum_{n=1}^{\infty} s_n(A) < \infty.$$

The class of all Hilbert–Schmidt operators on H is denoted by $\mathcal{H}(H)$ or $\mathcal{L}_{(2)}(H)$. The class of all nuclear operators on H is denoted by $\mathcal{N}(H)$ or $\mathcal{L}_{(1)}(H)$.

We recall that $s_n(A) \geqslant 0$. It is clear that $\mathcal{N}(H) \subset \mathcal{H}(H) \subset \mathcal{K}(H)$.

These two classes are the most important special cases of the Schatten classes $\mathcal{S}_p(H)$ defined by the condition $\{s_j(A)\} \subset l^p$.

There is an equivalent characterization of Hilbert–Schmidt operators that is often taken for the definition.

7.10.20. Theorem. (i) *An operator $A \in \mathcal{L}(H)$ on a separable Hilbert space H is a Hilbert–Schmidt operator precisely when for some orthonormal basis $\{e_n\}$ we have*

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty. \tag{7.10.1}$$

In this case such a series converges for every orthonormal basis and its sum does not depend on the basis.

(ii) *A bounded operator A on a separable Hilbert space H is a Hilbert–Schmidt operator precisely when A^* is a Hilbert–Schmidt operator. In addition,*

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{n=1}^{\infty} \|A^*e_n\|^2. \tag{7.10.2}$$

PROOF. Let us take the polar decomposition $A = U|A|$. If $A \in \mathcal{H}(H)$, then we have (7.10.1) for the eigenbasis of $|A|$. If (7.10.1) holds, then the operator A is compact. Indeed, the estimate $\bigl\|\sum_{n=1}^{\infty}(x, e_n)Ae_n\bigr\|^2 \leqslant \sum_{n=1}^{\infty} \|Ae_n\|^2\|x\|^2$ yields that the finite-dimensional operators $x \mapsto \sum_{n=1}^{N}(x, e_n)Ae_n$ converge to A with respect to the operator norm. We verify that the sum (7.10.1) does not depend on the basis. To this end we take an arbitrary orthonormal basis $\{\varphi_n\}$ and write the following equality:

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |(Ae_n, \varphi_k)|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |(e_n, A^*\varphi_k)|^2 = \sum_{k=1}^{\infty} \|A^*\varphi_k\|^2.$$

If $\{\psi_n\}$ is yet another basis, then the right-hand side equals $\sum_{n=1}^{\infty} \|A\psi_n\|^2$, since $A^{**} = A$. Hence the sum is independent of the basis. Applying this to the eigenbasis $\{\psi_n\}$ of the operator $|A|$, we obtain that $A \in \mathcal{H}(H)$. Assertion (i) is proved. On the way we have also proved (ii). $\square$

7.10.21. Proposition. *The class $\mathcal{H}(H)$ of all Hilbert–Schmidt operators on a separable Hilbert space H equipped with the inner product*

$$(A,B)_{\mathcal{H}} := \sum_{n=1}^{\infty}(Ae_n, Be_n) = \sum_{n=1}^{\infty}(B^*Ae_n, e_n)$$

is a separable Hilbert space. The corresponding Hilbert–Schmidt norm has the form

$$\|A\|_{\mathcal{H}} = \Bigl(\sum_{n=1}^{\infty}\|Ae_n\|^2\Bigr)^{1/2}.$$

If $A \in \mathcal{H}(H)$ and $B \in \mathcal{L}(H)$, then $AB \in \mathcal{H}(H)$, $BA \in \mathcal{H}(H)$ and

$$\|AB\|_{\mathcal{H}} \leqslant \|B\|_{\mathcal{L}(H)}\|A\|_{\mathcal{H}}, \quad \|BA\|_{\mathcal{H}} \leqslant \|B\|_{\mathcal{L}(H)}\|A\|_{\mathcal{H}}.$$

PROOF. It is clear from (7.10.1) that $\mathcal{H}(H)$ is a linear space. In addition, the series defining the inner product in $\mathcal{H}(H)$ converges absolutely, since $|(Ae_n, Be_n)| \leqslant \|Ae_n\|^2 + \|Be_n\|^2$. Let us verify the completeness of $\mathcal{H}(H)$. If a sequence of operators A_j is Cauchy in $\mathcal{H}(H)$, then it converges in norm to some operator $A \in \mathcal{L}(H)$, because $\|T\|_{\mathcal{L}(H)} \leqslant \|T\|_{\mathcal{H}}$. Let $\{e_n\}$ be an orthonormal basis. It is clear that $\sum_{n=1}^{N}\|Ae_n\|^2 \leqslant \sup_j \sum_{n=1}^{N}\|A_je_n\|^2 \leqslant \sup_j \|A_j\|_{\mathcal{H}}^2$ for all N, i.e., $A \in \mathcal{H}(H)$. Let us verify that $\|A - A_j\|_{\mathcal{H}} \to 0$. Let $\varepsilon > 0$. Find a number j_0 with $\|A_j - A_i\|_{\mathcal{H}}^2 \leqslant \varepsilon$ for all $i, j \geqslant j_0$. Let $m \geqslant j_0$. There is N with $\sum_{n=N+1}^{\infty}\|(A - A_m)e_n\|^2 \leqslant \varepsilon$, since $A, A_m \in \mathcal{H}(H)$. Finally, $\sum_{n=1}^{N}\|(A - A_m)e_n\|^2 \leqslant \varepsilon$, which is obtained by letting $i \to \infty$ in the estimate $\sum_{n=1}^{N}\|(A_i - A_m)e_n\|^2 \leqslant \varepsilon$ that holds for all $i \geqslant j_0$. The separability of $\mathcal{H}(H)$ follows from the fact that finite-dimensional operators are dense in $\mathcal{H}(H)$, since the operator $|A|$ with an eigenbasis $\{e_n\}$ is the limit with respect to the norm in $\mathcal{H}(H)$ of the finite-dimensional operators $x \mapsto (x, e_1)Ae_1 + \cdots + (x, e_n)Ae_n$, and finite-dimensional operators can be approximated by rational linear combinations of operators $x \mapsto (x, v_i)v_j$, where $\{v_i\}$ is a countable everywhere dense set. If $A \in \mathcal{H}(H)$ and $B \in \mathcal{L}(H)$, then $BA \in \mathcal{H}(H)$ and $\|BA\|_{\mathcal{H}} \leqslant \|B\|_{\mathcal{L}(H)}\|A\|_{\mathcal{H}}$, since $\|BAe_n\| \leqslant \|B\|_{\mathcal{L}(H)}\|Ae_n\|$. In addition, $AB \in \mathcal{H}(H)$, since we have the equalities $(AB)^* = B^*A^*$, $\|B^*\|_{\mathcal{L}(H)} = \|B\|_{\mathcal{L}(H)}$, $\|B^*\|_{\mathcal{H}} = \|B\|_{\mathcal{H}}$. □

It is useful to define Hilbert–Schmidt mappings between arbitrary Hilbert spaces.

7.10.22. Definition. *Let E_1 and E_2 be two Hilbert spaces. An operator $A \in \mathcal{L}(E_1, E_2)$ is called a Hilbert–Schmidt operator if the series $\sum_\alpha \|Ae_\alpha\|_{E_2}^2$ converges for some orthonormal basis $\{e_\alpha\}$ in E_1.*

The class of all Hilbert–Schmidt operators acting from E_1 to E_2 is denoted by $\mathcal{H}(E_1, E_2)$ or by $\mathcal{L}_{(2)}(E_1, E_2)$.

As above, it is easy to verify that the composition of two continuous operators between Hilbert spaces is a Hilbert–Schmidt operator if at least one of these operators has this property.

If the space E_1 is nonseparable, then the inclusion $A \in \mathcal{H}(E_1, E_2)$ means that A is zero on the orthogonal complement to some separable closed subspace

$E_0 \subset E_1$ and $\sum_{n=1}^{\infty} \|Ae_n\|_{E_2}^2 < \infty$ for some orthonormal basis $\{e_n\}$ in E_0. As above, one verifies that if the series of $\|Ae_\alpha\|_{E_2}^2$ converges for some orthonormal basis in E_1, then it converges for every orthonormal basis and the sum does not depend on the basis. The square root of the sum is called the Hilbert–Schmidt norm of the operator A and is denoted by $\|A\|_{\mathcal{H}}$. The Hilbert–Schmidt norm is generated by the inner product

$$(A, B) = \sum_{\alpha} (Ae_\alpha, Be_\alpha)_{E_2}.$$

This quantity does not depend on the orthonormal basis $\{e_\alpha\}$ in E_1.

The next result shows that abstract Hilbert–Schmidt operators are precisely integral operators on spaces L^2 with quadratically integrable kernels.

7.10.23. Proposition. (i) *Let μ be a nonnegative measure with values in $[0, +\infty]$ on a measurable space $(\Omega, \mathcal{A})$ and let $\mathcal{K}$ be a measurable function on $\Omega\times\Omega$ belonging to $L^2(\mu\otimes\mu)$. Then the operator defined on $L^2(\mu)$ by the formula*

$$Tx(t) = \int_\Omega \mathcal{K}(t, s)x(s)\, \mu(ds),$$

is a Hilbert–Schmidt operator. In addition,

$$\|T\|_{\mathcal{H}}^2 = \int_\Omega \int_\Omega |\mathcal{K}(t, s)|^2\, \mu(dt)\, \mu(ds) = \|\mathcal{K}\|_{L^2(\mu\otimes\mu)}^2.$$

(ii) *Every Hilbert–Schmidt operator on $L^2(\mu)$ has such a form. Hence every Hilbert–Schmidt operator is unitarily equivalent to an integral operator of the indicated form.*

PROOF. (i) The boundedness of the operator T with values in $L^2(\mu)$ is easily verified similarly to Example 6.9.4(iii). Let $\{e_n\}$ be an orthonormal sequence in $L^2(\mu)$. By Bessel's inequality applied to the functions $s \mapsto \mathcal{K}(t, s)$ we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \|Te_n\|^2 &= \sum_{n=1}^{\infty} \int_\Omega \left| \int_\Omega \mathcal{K}(t, s)e_n(s)\, \mu(ds) \right|^2 \mu(dt) \\ &\leqslant \int_\Omega \int_\Omega |\mathcal{K}(t, s)|^2\, \mu(ds)\, \mu(dt).\end{aligned}$$

Hence T is a Hilbert–Schmidt operator (in the separable case this is obvious, for the general case see Exercise 7.10.102). Let us take an orthonormal sequence $\{e_n\}$ such that the function $\mathcal{K}$ belongs to the closed linear span of the functions $\varphi_{n,m}(t, s) := e_n(t)e_m(s)$. Then for μ-a.e. t the function $s \mapsto \mathcal{K}(t, s)$ has an expansion in a series with respect to the elements e_n (Exercise 5.6.59). For such s in the relation above, in place of the inequality we have the Parseval equality, which gives $\sum_{n=1}^{\infty} \|Te_n\|^2 = \|\mathcal{K}\|_{L^2(\mu\otimes\mu)}^2$.

(ii) Let T be a Hilbert–Schmidt operator on $L^2(\mu)$ and let $\{e_\alpha\}$ be an orthonormal basis in $L^2(\mu)$. Then there is an at most countable part $\{e_n\}$ with $\|Te_n\| > 0$. We define an integral kernel $\mathcal{K}$ by the formula

$$\mathcal{K}(t, s) = \sum_{n,m\geqslant 1} (Te_n, e_m)e_n(t)e_m(s).$$

This double series converges in the space $L^2(\mu\otimes\mu)$, since the sequence of functions $e_n(t)e_m(s)$ is orthonormal and $\sum_{n,m\geqslant 1}|(Te_n,e_m)|^2=\sum_{n=1}^\infty\|Te_n\|^2<\infty$. The kernel $\mathcal{K}$ defines the operator $T_{\mathcal{K}}$ that coincides with T, because for every vector e_n in the chosen sequence we have $Te_n = T_{\mathcal{K}}e_n$ and for all other e_α we have $T_{\mathcal{K}}e_\alpha = 0 = Te_\alpha$. Finally, an abstract Hilbert–Schmidt operator is unitarily equivalent to some Hilbert–Schmidt operator on a suitable space $L^2(\mu)$ unitarily isomorphic to the original space (see Exercise 5.6.42). $\square$

From Theorem 7.10.15 we obtain at once the following result.

7.10.24. Proposition. *Let H_1 and H_2 be Hilbert spaces, $A,B\in\mathcal{L}(H_1,H_2)$, and $A(H_1)\subset B(H_1)$. If B is a Hilbert–Schmidt operator, then so is the operator A. If $H_1=H_2$ and B is a nuclear operator, then the operator A is also nuclear.*

Yet another important characterization of Hilbert–Schmidt operators describes them as absolutely summing and 2-summing operators.

A series $\sum_{n=1}^\infty x_n$ in a Banach space is called *unconditionally convergent* if it converges for all permutations of indices. If $\sum_{n=1}^\infty\|x_n\|<\infty$, then this series is called *absolutely convergent.*

7.10.25. Definition. *Let X and Y be two Banach spaces. An operator $T\in\mathcal{L}(X,Y)$ is called absolutely p-summing if for every weakly p-summable sequence $\{x_n\}\subset X$, i.e., satisfying the condition $\sum_{n=1}^\infty|l(x_n)|^p<\infty$ for all $l\in X^*$, we have $\sum_{n=1}^\infty\|Tx_n\|_Y^p<\infty$.*

The operator T is called absolutely summing if it takes unconditionally convergent series to absolutely convergent ones.

For any absolutely p-summing operator there is $C>0$ such that

$$\sum_{n=1}^\infty\|Tx_n\|_Y^p\leqslant C\sup_{\|l\|\leqslant 1}\sum_{n=1}^\infty|l(x_n)|^p \tag{7.10.3}$$

for every sequence $\{x_n\}\subset X$. Indeed, otherwise for each m there is a finite set $x_{m,1},\ldots,x_{m,k}$ with $\sup_{\|l\|\leqslant 1}\sum_{i=1}^k|l(x_{m,i})|^p\leqslant 2^{-m}$ and $\sum_{i=1}^k\|Tx_{m,i}\|_Y^p\geqslant 1$, which leads to a contradiction. The smallest possible C is denoted by $\pi_p(T)$.

These classes are stable under compositions from the right and left with bounded operators. According to the Dvoretzky–Rogers theorem, in every infinite-dimensional Banach space X there is a conditionally convergent series that is not absolutely convergent. If X has no subspaces isomorphic to c_0 (and only in this case), then the unconditional convergence series of x_n is equivalent to the weak 1-summability of $\{x_n\}$ (see [**303**, Chapters 3, 4]).

7.10.26. Proposition. *Let E_1 and E_2 be two Hilbert spaces. An operator $T\in\mathcal{L}(E_1,E_2)$ is a Hilbert–Schmidt operator precisely when is it absolutely 2-summing or absolutely summing.*

PROOF. Let T be an absolutely 2-summing operator and $\{e_\alpha\}$ an orthonormal basis in E_1. For every $y\in E_1$ we have $\sum_\alpha|(y,e_\alpha)|^2=\|y\|^2<\infty$, whence

$\sum_\alpha \|Te_\alpha\|^2 < \infty$. Suppose now that $T \in \mathcal{H}(E_1, E_2)$ and $\{h_n\}$ is a sequence in E_1 with $\sum_{n=1}^\infty |(h_n, y)|^2 < \infty$ for all $y \in E_1$. We can assume that E_1 is the closure of the linear span of the vectors h_n. If E_1 is finite-dimensional, then the assertion is trivial. Hence we assume that E_1 is infinite-dimensional and take a basis $\{e_n\}$. Let us define an operator S on the space E_1 by $Se_n = h_n$, i.e., $Sx = \sum_{n=1}^\infty (x, e_n)h_n$. This series converges weakly in E_1, since for every $y \in E_1$ the series $\sum_{n=1}^\infty (x, e_n)(h_n, y)$ converges absolutely, which follows from convergence of the series of $|(x, e_n)|^2$ and $|(h_n, y)|^2$. Hence the operator S is continuous. This shows that TS is a Hilbert–Schmidt operator. Therefore, the series of $\|Th_n\|^2 = \|TSe_n\|^2$ converges. If T is absolutely summing and $\{e_n\}$ is an orthonormal sequence, then for any $(x_n) \in l^2$ the series of $x_n e_n$ converges unconditionally. This yields convergence of the series of $|x_n|\|Te_n\|$, whence $\{\|Te_n\|\} \in l^2$. Hence $T \in \mathcal{H}(E_1, E_2)$. The converse follows from Exercise 7.10.130. We observe that this proposition is true for all $p > 0$ in place of 2 or 1. □

7.10.27. Theorem. *Suppose that μ is a Borel probability measure on a topological space Ω. Then the identity embedding $j\colon C_b(\Omega) \to L^2(\mu)$ is absolutely 2-summing.*

The same is true if for Ω we take an arbitrary probability space and replace $C_b(\Omega)$ by $L^\infty(\mu)$.

PROOF. Let $\{x_n\} \subset C_b(\Omega)$ and $\sum_{n=1}^\infty |l(x_n)|^2 < \infty$ for all $l \in C_b(\Omega)^*$. Then the sequence of operators $T_n\colon C_b(\Omega)^* \to l^2$, $l \mapsto \bigl(l(x_1), \ldots, l(x_n), 0, 0, \ldots\bigr)$, is pointwise bounded. By the Banach–Steinhaus theorem

$$\sup_{\|l\| \leqslant 1} \sum_{n=1}^\infty |l(x_n)|^2 \leqslant C < \infty.$$

Taking for l the functionals $x \mapsto x(t)$, we obtain $\sum_{n=1}^\infty |x_n(t)|^2 \leqslant C$. Thus, $\sum_{n=1}^\infty \|j(x_n)\|_2^2 \leqslant C$. The case of $L^\infty(\mu)$ for a general measure space reduces to the case of $C(K)$ with a compact space K. For this we need the following fact from §11.7(i): there exists a compact space K such that the algebra $L^\infty(\mu)$ is linearly isometric to the algebra $C(K)$ and on K there is a probability Borel measure ν such that the indicated isomorphism between $L^\infty(\mu)$ and $C(K)$ defines an isomorphism between $L^2(\mu)$ and $L^2(\nu)$. □

On account of Proposition 7.10.26 we obtain the following interesting fact.

7.10.28. Corollary. *Let H be a Hilbert space, $(\Omega, \mathcal{A}, \mu)$ a probability space, and $A\colon H \to L^2(\mu)$ a continuous operator such that $A(H) \subset L^\infty(\mu)$. Then A is a Hilbert–Schmidt operator.*

This corollary admits an important reinforcement that is a particular case of a more general result due to V. B. Korotkov.

7.10.29. Theorem. *Let μ be a probability measure on a measurable space $(\Omega, \mathcal{A})$. A bounded operator T on $L^2(\mu)$ is a Hilbert–Schmidt operator precisely when there exists a nonnegative function $\Phi \in \mathcal{L}^2(\mu)$ such that, for every function*

$x \in L^2(\mu)$, the element Tx has a modification which satisfies the estimate

$$|Tx(\omega)| \leqslant C_x \Phi(\omega) \tag{7.10.4}$$

with some number $C_x \geqslant 0$.

PROOF. The necessity of this condition is clear from the facts established above and the estimate

$$|Tx(t)| \leqslant \|x\| \biggl(\int_\Omega |\mathcal{K}(t,s)|^2 \, \mu(ds) \biggr)^{1/2}$$

for the operator T defined by the kernel $\mathcal{K} \in \mathcal{L}^2(\mu \otimes \mu)$, since on the right we have a function from $\mathcal{L}^2(\mu)$. Suppose that (7.10.4) is fulfilled. Replacing Φ by $\Phi + 1$, we can assume that $\Phi \geqslant 1$. We observe that for C_x we can take $C\|x\|$ with some $C > 0$. Indeed, the bounded operator $S = \Phi^{-1}T$ takes values in $L^\infty(\mu)$ and hence is bounded as an operator from $L^2(\mu)$ to $L^\infty(\mu)$ (see Corollary 6.2.8), which gives the desired number C. Set

$$T_\varepsilon x(\omega) := |\varepsilon \Phi(\omega) + 1|^{-1} Tx(\omega), \quad \varepsilon > 0.$$

Then $|T_\varepsilon x(\omega)| \leqslant C\varepsilon^{-1}\|x\|$, i.e., the range of the operator T_ε is contained in $L^\infty(\mu)$. According to the previous corollary the operator T_ε is a Hilbert–Schmidt operator. For every $x \in L^2(\mu)$ we have $\lim\limits_{\varepsilon \to 0} T_\varepsilon x = Tx$. Hence it suffices to establish the uniform boundedness of the Hilbert–Schmidt norms of the operators T_ε. Since by the Lebesgue dominated convergence theorem

$$\bigl\| \Phi(\varepsilon\Phi + 1)^{-1} \bigr\|_{L^2(\mu)} \to \|\Phi\|_{L^2(\mu)} \quad \text{as } \varepsilon \to 0,$$

it suffices to show that for a bounded function Φ one has $\|T\|_{\mathcal{H}} \leqslant C\|\Phi\|_{L^2(\mu)}$. We already know that in case of a bounded function Φ the operator T is a Hilbert–Schmidt operator, hence is defined by some kernel $\mathcal{K} \in \mathcal{L}^2(\mu \otimes \mu)$. Hence for every function $x \in \mathcal{L}^2(\mu)$ with $\|x\|_{L^2(\mu)} \leqslant 1$ we have

$$|Tx(t)| = \biggl| \int_\Omega \mathcal{K}(t,s)x(s) \, \mu(ds) \biggr| \leqslant C\Phi(t) \tag{7.10.5}$$

for μ-a.e. t. However, the corresponding measure zero set can depend on x. By the compactness of T there is a separable closed subspace $H \subset L^2(\mu)$ with $T(H) \subset H$ and $T(H^\perp) = 0$. Let us take a countable set $\{x_n\}$ dense in the unit ball of H. This enables us to pass to the case of a separable space $L^2(\mu)$ (for which it suffices to consider the measure μ on the σ-algebra generated by all functions x_n and Φ). Then (7.10.5) is fulfilled almost everywhere simultaneously for all x_n. By our choice of $\{x_n\}$ the quantity $\sup\limits_n \Bigl| \int_\Omega \mathcal{K}(t,s)x_n(s) \, \mu(ds) \Bigr|$ is the L^2-norm of the function $s \mapsto \mathcal{K}(t,s)$ for points t such that this function belongs to $L^2(\mu)$. Therefore, for μ-a.e. t we have

$$\int_\Omega |\mathcal{K}(t,s)|^2 \, \mu(ds) \leqslant C^2 \Phi(t)^2.$$

Hence $\|T\|_{\mathcal{H}} \leqslant \|\mathcal{K}\|_{L^2(\mu\otimes\mu)} \leqslant C^2\|\Phi\|_{L^2(\mu)}$, as required. $\square$

The following theorem due to Pietch shows that in case of a general Banach space absolutely 2-summing operators are also connected with the space L^2.

7.10.30. Theorem. *Let X and Y be Banach spaces, B^* the closed unit ball in the space X^* equipped with the $*$-weak topology, $\sigma\bigl(C(B^*)\bigr)$ the σ-algebra generated by continuous functions on the compact space B^*. An operator $T \in \mathcal{L}(X,Y)$ is absolutely 2-summing precisely when there exists a bounded nonnegative measure ν on $\sigma\bigl(C(B^*)\bigr)$ such that*

$$\|Tx\|_Y^2 \leqslant \int_{B^*} |\xi(x)|^2\, \nu(d\xi).$$

PROOF. If such a measure exists, then for all $x_1, \ldots, x_n \in X$ we have

$$\sum_{i=1}^n \|Tx_i\|_Y^2 \leqslant \int_{B^*} \sum_{i=1}^n |\xi(x_i)|^2\, \nu(d\xi) \leqslant \nu(B^*) \sup_{\|\xi\|\leqslant 1} \sum_{i=1}^n |\xi(x_i)|^2,$$

whence $\pi_2(T) \leqslant \nu(B^*)$. Conversely, suppose that $\pi_2(T) = 1$. Let us consider the following subsets in the Banach space $C(B^*)$:

$$F_1 := \{f \in C(B^*)\colon\ \sup\nolimits_{\|\xi\|\leqslant 1} f(\xi) < 1\}$$

and F_2 equal to the convex envelope of the set of functions $f \in C(B^*)$ of the form $f(\xi) = |\xi(x)|^2$, where $\|Tx\|_Y = 1$. These sets are convex, F_1 is open and $F_1 \cap F_2 = \varnothing$, since $\pi_2(T) = 1$. By the Hahn–Banach theorem and the Riesz theorem on the representation of functionals on $C(B^*)$ by measures, there exists a measure ν on the aforementioned σ-algebra in B^* such that

$$\sup_{f\in F_1} \int_{B^*} f(\xi)\, \nu(d\xi) \leqslant \inf_{f\in F_2} \int_{B^*} f(\xi)\, \nu(d\xi).$$

Since F_1 contains all negative functions, the measure ν is nonnegative. We can assume that $\nu(B^*) = 1$. Then the left-hand side of the previous estimate equals 1. Whenever $\|Tx\|_Y = 1$, for the function $f_x(\xi) := |\xi(x)|^2$ we obtain $f_x \in F_2$, hence the integral of f_x with respect to the measure ν is not less than 1, which yields the desired inequality. □

For example, from this theorem one can derive Theorem 7.10.27, taking for ν the image of the measure μ under the embedding $\Omega \subset C_b(\Omega)^*$, $\omega \mapsto \delta_\omega$. Then the integral of $|x(\omega)|^2$ with respect to the measure μ will equal the integral of $|\delta_\omega(x)|^2$ with respect to the measure μ, i.e., the integral of $|\xi(x)|^2$ with respect to the measure ν.

Diagonal operators are defined in Example 6.1.5(vii). Let us mention an interesting theorem due to von Neumann (see [**252**, Theorem 14.13]).

7.10.31. Theorem. *For every selfadjoint operator A on a separable Hilbert space H and every number $\varepsilon > 0$, there exists a diagonal selfadjoint operator D and a selfadjoint Hilbert–Schmidt operator S_ε on the space H such that $\|S_\varepsilon\|_{\mathcal{H}} \leqslant \varepsilon$ and $A = D + S_\varepsilon$.*

We now return to nuclear operators and their useful connections with Hilbert–Schmidt operators.

7.10.32. Remark. Let $A \in \mathcal{L}(H)$ be such that for some C one has

$$\sum_{n=1}^{\infty} |(A\psi_n, \varphi_n)| \leqslant C$$

for all pairs of orthonormal bases $\{\psi_n\}$ and $\{\varphi_n\}$. Then $A \in \mathcal{N}(H)$. Indeed, Proposition 7.8.12 yields that the operator A is compact. Let us take its polar decomposition $A = U|A|$ and the eigenbasis $\{\psi_n\}$ for the operator $|A|$. For $\{\varphi_n\}$ we take the orthonormal sequence of all nonzero vectors $U\psi_n$ complemented to an orthonormal basis. This gives convergence of the series of $s_n(A)$.

7.10.33. Theorem. *Let H be a separable Hilbert space and $A \in \mathcal{L}(H)$. The following conditions are equivalent*:

(i) $A \in \mathcal{N}(H)$; (ii) $A^* \in \mathcal{N}(H)$; (iii) $A = A_1A_2$, *where* $A_1, A_2 \in \mathcal{H}(H)$;

(iv) *there exist two sequences of vectors $\{v_k\}$ and $\{u_k\}$ with $\|u_k\| = \|v_k\| = 1$ and a scalar sequence $\{\lambda_k\} \in l^1$ such that*

$$Ax = \sum_{k=1}^{\infty} \lambda_k (x, u_k) v_k; \tag{7.10.6}$$

(v) *there is an orthonormal basis $\{e_n\}$ with $\sum_{n=1}^{\infty} \|Ae_n\| < \infty$.*

PROOF. The equivalence of (i) and (ii) follows from the fact that according to Exercise 7.10.65 the operators A^*A and AA^* have the same nonzero eigenvalues (for zero this can be false). Hence for all nonzero eigenvalues $s_j(A) = s_j(A^*)$. If A is nuclear, then $|A|^{1/2}$ is a Hilbert–Schmidt operator, hence the polar decomposition $A = U|A|$ gives the representation $A = U|A|^{1/2}|A|^{1/2}$, where $U|A|^{1/2} \in \mathcal{H}(H)$, i.e., we obtain (iii).

Let us derive (iv) from (iii). Suppose we have a representation $A = A_1A_2$, where $A_1, A_2 \in \mathcal{H}(H)$. Let us take the polar decomposition $A_2 = V|A_2|$. The operator $|A_2|$ has an eigenbasis $\{e_n\}$ and eigenvalues $\{\alpha_n\}$. Hence we have $Ax = \sum_{n=1}^{\infty} \alpha_n (x, e_n) A_1 V e_n$. If $\beta_n := \|A_1Ve_n\| > 0$, we set $v_n := \beta_n^{-1} A_1 V e_n$. Then for the numbers $\lambda_n := \alpha_n\beta_n$ we obtain $\{\lambda_n\} \in l^1$, which gives (7.10.6) with $u_n = e_n$. Suppose now that (iv) holds. Then A is obviously compact. If $\{\psi_n\}$ and $\{\varphi_n\}$ are two orthonormal bases, then the series of $|(A\psi_n, \varphi_n)|$ is estimated by the double series of $|\lambda_k(\psi_n, u_k)(v_k, \varphi_n)|$, which is dominated by the series of $|\lambda_k|$, since $|(\psi_n, u_k)(v_k, \varphi_n)| \leqslant [|(\psi_n, u_k)|^2 + |(v_k, \varphi_n)|^2]/2$ and the sum over n in the right-hand side equals 1. According to Remark 7.10.32 the operator A is nuclear. Moreover, (v) is fulfilled for the eigenbasis of $|A|$. Finally, (v) implies (iv), because we take $\lambda_k = \|Ae_k\|$, $u_k = e_k$ and $v_k = \|Ae_k\|^{-1} Ae_k$ if $Ae_k \neq 0$. □

Note that the series in (v) converges for some, but in general not for every orthonormal basis (Exercise 7.10.115). For a basis for which convergence holds we can take the eigenbasis of $|A|$. Then for the expansion (7.10.6) we can take $Ax = \sum_{n=1}^{\infty} s_n(A)(x, e_n) U e_n$, where only e_n with $Ue_n \neq 0$ are taken into account.

The next result enables us to introduce the trace of any nuclear operator.

7.10.34. Lemma. *Let $A \in \mathcal{N}(H)$ and let $\{\varphi_n\}$ be an orthonormal basis in H. If A has the form* (7.10.6)*, then*

$$\sum_{n=1}^{\infty} \lambda_n (v_n, u_n) = \sum_{n=1}^{\infty} (A\varphi_n, \varphi_n), \tag{7.10.7}$$

the series in the right-hand side converges absolutely and its sum does not depend on our choice of the basis. In particular,

$$\sum_{n=1}^{\infty} (A\varphi_n, \varphi_n) = \sum_{n=1}^{\infty} s_n(A)(Ue_n, e_n),$$

where $\{e_n\}$ is the eigenbasis of the operator $|A|$.

PROOF. The series in the left-hand side of (7.10.7) converges absolutely and

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n (v_n, u_n) &= \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\infty} (v_n, \varphi_k)(\varphi_k, u_n) \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \lambda_n (v_n, \varphi_k)(\varphi_k, u_n) = \sum_{k=1}^{\infty} (A\varphi_k, \varphi_k),\end{aligned}$$

which shows convergence of the series in the right-hand side. Its absolute convergence follows from the fact that the sum does not change under permutations of terms in this series (note that any permutation of $\{\varphi_k\}$ remains an orthonormal basis). □

The *trace of an operator* $A \in \mathcal{N}(H)$ is defined by

$$\operatorname{tr} A := \sum_{k=1}^{\infty} (A\varphi_k, \varphi_k),$$

where $\{\varphi_k\}$ is an orthonormal basis. The lemma shows that the trace is well-defined. The following fact is true.

7.10.35. Proposition. *Let H be a complex Hilbert space. An operator $A \in \mathcal{L}(H)$ is nuclear precisely when for every orthonormal basis $\{e_n\}$ in H the series $\sum_{n=1}^{\infty} (Ae_n, e_n)$ converges.*

PROOF. Let us show that $A \in \mathcal{N}(H)$ if all such series converge. If $A \geqslant 0$, then it suffices to have convergence for some basis, since in that case $A^{1/2}$ is a Hilbert–Schmidt operator. If A is selfadjoint, then there exist closed subspaces H_1 and H_2 such that $H_1 \perp H_2$, $H = H_1 \oplus H_2$, $A(H_i) \subset H_i$. This follows from the theorem about representation of A as the multiplication by a real function φ on $L^2(\mu)$: for H_1 and H_2 we can take the subspaces of functions vanishing outside $\{\varphi > 0\}$ and $\{\varphi \leqslant 0\}$. The operators $A|_{H_1}$ and $-A|_{H_2}$ are nonnegative. Let us take orthonormal bases $\{\varphi_n\}$ and $\{\psi_n\}$ in H_1 and H_2. Let $e_n = \varphi_n$ for odd n and $e_n = \psi_n$ for even n. Then both series $\sum_{n=1}^{\infty} (A\varphi_n, \varphi_n)$ and $\sum_{n=1}^{\infty} (A\psi_n, \psi_n)$ must converge separately, since the series of (Ae_n, e_n) converges for every permutation of its terms. Therefore, the restrictions of A to H_1 and H_2 are nuclear operators, whence $A \in \mathcal{N}(H)$. Finally, in the general case we observe that the operator A^* satisfies the same condition as A. Then this condition is fulfilled for the selfadjoint operators $B_1 = A + A^*$ and $B_2 = i(A - A^*)$. As we have shown, $B_1, B_2 \in \mathcal{N}(H)$. Hence $A = (B_1 - iB_2)/2 \in \mathcal{N}(H)$. $\square$

Note that here it is not enough to have convergence of only one such series (even absolute): see Exercise 7.10.116. For real spaces this proposition is false (it suffices to take a noncompact operator A with $(Ax, x) = 0$).

7.10.36. Theorem. *Let H be a complex or real Hilbert space.*

(i) *Let $A \in \mathcal{N}(H)$. Then*

$$\operatorname{tr}|A| = \sup \sum_{n=1}^{\infty} |(A\psi_n, \varphi_n)|, \tag{7.10.8}$$

where sup *is taken over all pairs of orthonormal bases $\{\psi_n\}$ and $\{\varphi_n\}$.*

(ii) *Let $A \in \mathcal{N}(H)$. Then*

$$\operatorname{tr}|A| = \inf \sum_{n=1}^{\infty} |\lambda_n|, \tag{7.10.9}$$

where inf *is taken over all representations of the form* (7.10.6).

(iii) *The space of nuclear operators on a separable Hilbert space H is a separable Banach space with respect to the norm $\|A\|_{(1)} := \operatorname{tr}|A|$.*

(iv) *An operator $A \in \mathcal{L}(H)$ is nuclear precisely when so is A^*. In addition, $\|A\|_{(1)} = \|A^*\|_{(1)}$ and $\operatorname{tr} A = \overline{\operatorname{tr} A^*}$.*

(v) *If $A \in \mathcal{N}(H)$ and $T \in \mathcal{L}(H)$, then $AT, TA \in \mathcal{N}(H)$ and*

$$\operatorname{tr} AT = \operatorname{tr} TA \quad \textit{and} \quad \|AT\|_{(1)} \leqslant \|T\|_{\mathcal{L}(H)} \|A\|_{(1)}.$$

PROOF. We shall assume that H is complex (the real case is similar). (i) Let us take the polar decomposition $A = U|A|$ and the eigenbasis $\{e_n\}$ of the operator $|A|$. Then $A\psi_n = \sum_{j=1}^{\infty} s_j(A)(\psi_n, e_j)Ue_j$, whence

$$\sum_{n=1}^{\infty} (A\psi_n, \varphi_n) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} s_j(A)(\psi_n, e_j)(Ue_j, \varphi_n),$$

which is estimated by $\operatorname{tr}|A|$ in absolute value, as in the proof of Theorem 7.10.33. Since the same is true when we replace e_n by $e^{i\theta_n}e_n$ with arbitrary real θ_n, the right-hand side of (7.10.8) is not greater than the left-hand side. On the other hand, all nonzero vectors Ue_n form an orthonormal system. Complementing it to a basis $\{\varphi_n\}$, we obtain a pair of bases for which the equality is achieved.

(ii) We have seen in the proof of Theorem 7.10.33 that the sum of the series of $s_n(A)$ does not exceed the sum of the series of $|\lambda_n|$ for every representation (7.10.6). The equality is achieved for $Ax = \sum_{n=1}^{\infty} s_n(A)(x, e_n)Ue_n$, where $A = U|A|$ is the polar decomposition and $\{e_n\}$ is the eigenbasis for $|A|$.

(iii) It is clear from assertion (iv) of Theorem 7.10.33 that $\mathcal{N}(H)$ is a linear space. Estimate (7.10.8) along with the obvious inequality $\|A\| \leqslant \|A\|_{(1)}$ shows that $\|\cdot\|_{(1)}$ is a norm. If a sequence $\{A_n\}$ is Cauchy with respect to this norm, then it converges to some operator $A \in \mathcal{K}(H)$ in the operator norm. Let $A = U|A|$ be the polar decomposition of A and $\{e_n\}$ the eigenbasis of $|A|$. Set $\psi_n = e_n$

and $\varphi_n = Ue_n$ if $Ue_n \neq 0$. Let us complement $\{\varphi_n\}$ to an orthonormal basis. We obtain convergence of the series of $s_n(A)$, since the series of $|(A\psi_n, \varphi_n)|$ converges by (7.10.8) and the uniform boundedness of the norms $\|A_n\|_{(1)}$. We show that $\|A - A_n\|_{(1)} \to 0$. Let $\varepsilon > 0$. Find a number n_0 with $\|A_m - A_n\|_{(1)} \leqslant \varepsilon$ for all $n, m \geqslant n_0$. Suppose that we have two orthonormal bases $\{\psi_k\}$ and $\{\varphi_k\}$ in the space H. Let $m \geqslant n_0$. Find N with $\sum_{k=N+1}^{\infty} |(A\psi_k - A_m\psi_k, \varphi_k)| \leqslant \varepsilon$. The bound $\sum_{k=1}^{N} |(A_j\psi_k - A_m\psi_k, \varphi_k)| \leqslant \varepsilon$ for $j \geqslant n_0$ yields in the limit the same bound for A in place of A_j. Thus, by (7.10.8) we obtain that $\|A - A_m\|_{(1)} \leqslant 2\varepsilon$. It remains to observe that every operator $A \in \mathcal{N}(H)$ is approximated with respect to the norm $\|\cdot\|_{(1)}$ by finite-dimensional operators, since this is true for $|A|$ (it suffices to use the eigenbasis of $|A|$).

The first assertion in (iv) follows from the fact that $|A|$ and $|A^*|$ have the same nonzero eigenvalues, which has already been noted above. The same reasoning gives the equality $\|A\|_{(1)} = \|A^*\|_{(1)}$. The last equality in (iv) is obvious.

In assertion (v) the nuclearity of the operators AT and TA is obvious from assertion (iv) of Theorem 7.10.33 (one can also use the connection with Hilbert–Schmidt operators). The estimate $\|AT\|_{(1)} \leqslant \|T\|_{\mathcal{L}(H)}\|A\|_{(1)}$ follows from assertion (ii) and representation (7.10.6). Let us verify the equality of the traces of the operators AT and TA. Writing the operator A in the form (7.10.6), we arrive at the following two equalities:

$$TAx = \sum_{n=1}^{\infty} \lambda_n(x, u_n)Tv_n, \ ATx = \sum_{n=1}^{\infty} \lambda_n(Tx, u_n)v_n = \sum_{n=1}^{\infty} \lambda_n(x, T^*u_n)v_n.$$

Equality (7.10.7) shows that the trace of TA equals $\sum_{n=1}^{\infty} \lambda_n(Tv_n, u_n)$ and the trace of AT equals $\sum_{n=1}^{\infty} \lambda_n(v_n, T^*u_n)$, i.e., the same number. □

Let us give a sufficient condition for an operator on $L^2[0,1]$ to be nuclear.

7.10.37. Example. Let T be a bounded operator on $L^2[0,1]$ such that its range is contained in the set of Lipschitz functions (or, more generally, there exists a function $\Phi \in L^2[0,1]$ such that the range of T is contained in the class of absolutely continuous functions x for which $|x'(t)| \leqslant C_x|\Phi(t)|$ a.e.). Then T is a nuclear operator.

PROOF. First, we suppose additionally that all functions from the range of T equal zero at 0. The operator $Sx = (Tx)'$ takes values in $L^\infty[0,1]$. According to Corollary 7.10.28 the operator S is a Hilbert–Schmidt operator. In the case of the second more general condition we apply Theorem 7.10.29. The Volterra operator V of indefinite integration is also a Hilbert–Schmidt operator. Hence $T = VS$ is nuclear. In the general case we observe that the operator T is continuous as an operator with values in $C[0,1]$. Hence the functional $l\colon\ x \mapsto Tx(0)$ is continuous. This enables us to write T in the form $Tx = l(x)1 + T_0x$, where T_0 has the form considered above. Then T is nuclear, since so are T_0 and the one-dimensional operator $x \mapsto l(x)1$. □

An important particular case is the following integral operator with an integral kernel satisfying the Lipschitz condition in the first variable.

7.10.38. Example. Suppose that a measurable function $\mathcal{K}$ belongs to $\mathcal{L}^2$ on the square $[0,1]\times[0,1]$ and that for almost every s the function $t\mapsto\mathcal{K}(t,s)$ satisfies the Lipschitz condition with a constant $\Phi(s)$ such that the function Φ belongs to $\mathcal{L}^2[0,1]$. Then the operator

$$Tx(t)=\int_0^1 \mathcal{K}(t,s)x(s)\,ds$$

is nuclear. The same is true under the following weaker condition: for almost every point s the function $t\mapsto\mathcal{K}(t,s)$ is absolutely continuous on the interval $[0,1]$ and $\partial\mathcal{K}(t,s)/\partial t\in\mathcal{L}^2([0,1]\times[0,1])$.

Note that the continuity of the kernel $\mathcal{K}$ is not sufficient for the nuclearity of the integral operator with the kernel $\mathcal{K}$. T. Carleman gave the following example: $\mathcal{K}(t,s)=F(t-s)$, where F is a continuous function with a period 1 and the Fourier expansion $F(t)=\sum_{k\in\mathbb{Z}}c_k e^{2\pi ikt}$ such that $\sum_k|c_k|=\infty$. Then the functions $e_k(t)=e^{2\pi ikt}$ constitute an orthonormal eigenbasis for the integral operator with the kernel $\mathcal{K}$ and $\{c_k\}$ is the sequence of the corresponding eigenvalues. However, if it is known in addition that the integral operator with the kernel $\mathcal{K}$ is nonnegative (its quadratic form is nonnegative), then it must be nuclear.

7.10.39. Example. Let $\mathcal{K}$ be a continuous real function on $[0,1]\times[0,1]$ with $\mathcal{K}(t,s)=\mathcal{K}(s,t)$ such that the integral operator T given on $L^2[0,1]$ by the kernel $\mathcal{K}$ is nonnegative. Then T is a nuclear operator and

$$\operatorname{tr}T=\int_0^1\mathcal{K}(t,t)\,dt.$$

Indeed, let us take the eigenbasis $\{e_n\}$ of the operator T with eigenvalues λ_n. By Mercer's theorem 7.10.43 proved below the series $\sum_{n=1}^\infty\lambda_n e_n(t)e_n(s)$ converges to $\mathcal{K}(t,s)$ uniformly on the square. In particular, it converges uniformly on the diagonal, which after integration gives the indicated equality. Note that T is nuclear if in place of the continuity of $\mathcal{K}$ we assume only the measurability and boundedness (of course, keeping the condition $T\geqslant 0$). This follows from the result above, since one can take a sequence of continuous kernels $\mathcal{K}_n$ with $\sup|\mathcal{K}_n(t,s)|\leqslant\sup|\mathcal{K}(t,s)|$ converging in measure to $\mathcal{K}$ and generating nonnegative operators (see Gokhberg, Krein [**226**, p. 149–151]).

The next theorem describes an interesting connection between nuclear operators and functionals on the space of operators.

7.10.40. Theorem. (i) *For every $S\in\mathcal{N}(H)$ the functional*

$$K\mapsto\operatorname{tr}SK$$

on $\mathcal{K}(H)$ has the norm $\|S\|_{(1)}$. Conversely, every continuous functional on $\mathcal{K}(H)$ admits such a representation, i.e., the space $\mathcal{K}(H)^*$ is naturally isomorphic to the space $\mathcal{N}(H)$.

(ii) *For every $T\in\mathcal{L}(H)$ the functional*

$$S\mapsto\operatorname{tr}TS$$

on $\mathcal{N}(H)$ has the norm $\|T\|_{\mathcal{L}(H)}$. Conversely, every continuous functional on $\mathcal{N}(H)$ admits such a representation, i.e., the space $\mathcal{N}(H)^*$ is naturally isomorphic to the space $\mathcal{L}(H)$.

PROOF. (i) We already know that the norm $\|\Lambda\|$ of the indicated functional Λ does not exceed $\|S\|_{(1)}$. We show that $\|\Lambda\| \geqslant \|S\|_{(1)}$. Let $\varepsilon > 0$. Let us take the polar decomposition $S = U|S|$ and the eigenbasis $\{e_n\}$ of the operator $|S|$ with eigenvalues s_n. Then

$$\|S\|_{(1)} \leqslant \sum_{n=1}^{N} s_n + \varepsilon$$

for some N, where $s_n > 0$. Let us take a finite-dimensional operator K_N with $K_N U e_n = e_n$, $i = 1, \ldots, N$ and extend it by zero on the orthogonal complement of the linear span of $e_1, \ldots, e_N$. Then $\|K_N\|_{\mathcal{L}(H)} = 1$ and

$$\operatorname{tr} K_N S = \sum_{n=1}^{N} s_n (K_N U e_n, e_n) = \sum_{n=1}^{N} s_n \geqslant \|S\|_{(1)} - \varepsilon,$$

which gives the desired bound. Thus, we have an isometric embedding of the space $\mathcal{N}(H)$ into $\mathcal{K}(H)^*$. Let $\Lambda \in \mathcal{K}(H)^*$. For every $u, v \in H$, we have a one-dimensional operator $K_{u,v}(x) := (x, u)v$. Set

$$B(u, v) := \Lambda(K_{u,v}).$$

The function B is linear in u and conjugate-linear in v. The continuity of B is clear from the equality $\|K_{u,v}\|_{(1)} = \|u\|\,\|v\|$ (see Exercise 7.10.63). According to Exercise 7.2.1 there exists an operator $S \in \mathcal{L}(H)$ with $(Su, v) = B(u, v)$. We show that this is the required operator. For any two infinite-dimensional orthonormal sequences $\{\psi_n\}$ and $\{\varphi_n\}$ and every element $(\xi_n) \in c_0$, the series $\sum_{n=1}^{\infty} \xi_n K_{\psi_n, \varphi_n}$ converges with respect to the operator norm to some operator $K \in \mathcal{K}(H)$. By the continuity of Λ the series of $\xi_n \Lambda(K_{\psi_n,\varphi_n}) = \xi_n (S\psi_n, \varphi_n)$ converges and the absolute value of the sum does not exceed $\|\Lambda\|\,\|K\|_{\mathcal{L}(H)}$. It is readily seen that $\|K\|_{\mathcal{L}(H)} = \sup_n |\xi_n|$. Therefore, $\{(S\psi_n, \varphi_n)\} \in l^1$ and $\sum_{n=1}^{\infty} |(S\psi_n, \varphi_n)| \leqslant \|\Lambda\|$, which gives the estimate $\|S\|_{(1)} \leqslant \|\Lambda\|$ by the previous remark. Assertion (ii) is proved similarly. □

This theorem yields the equality $\mathcal{K}(H)^{**} = \mathcal{L}(H)$.

One of the deepest results on operator traces is the following theorem due to V. B. Lidskii. Its rather difficult proof can be read in several books, for example, see Gokhberg, Krein [**226**, Chapter III, §8] (the shortest proof is presented in the book Simon [**554**]).

7.10.41. Theorem. *For every $A \in \mathcal{N}(H)$ we have*

$$\operatorname{tr} A = \sum_{n=1}^{\infty} \lambda_n,$$

where λ_n are all eigenvalues of A counted with multiplicities and in the case of absence of eigenvalues the right-hand side is defined to be zero.

The most difficult part of the proof concerns precisely the case of absence of eigenvalues. It is far from being obvious that in this case the trace vanishes. Nonzero eigenvalues are considered in Exercise 7.10.106. About eigenvalues of compact operators, see also König [**333**] and Pietsch [**484**].

7.10(v). Integral operators and Mercer's theorem

Let us mention several interesting facts connected with the Hilbert–Schmidt theorem and useful in the study of integral equations. Let $\mathcal{K} \in \mathcal{L}^2([0,1]\times[0,1])$ be a complex function with $\mathcal{K}(t,s) = \overline{\mathcal{K}(s,t)}$. Let us consider the compact selfadjoint operator

$$T_{\mathcal{K}} x(t) = \int_0^1 \mathcal{K}(t,s)x(s)\,ds$$

on $L^2[0,1]$. By the Hilbert–Schmidt theorem there exists an orthonormal basis $\{e_n\}$ with $T_{\mathcal{K}} e_n = \lambda_n e_n$, where $\lambda_n \in \mathbb{R}^1$. Hence for every n we have

$$\lambda_n e_n(t) = \int_0^1 \mathcal{K}(t,s)e_n(s)\,ds$$

for almost all t. Hence for almost all t this equality is true simultaneously for all n. In addition, for almost all t the function $s \mapsto \mathcal{K}(s,t)$ belongs to $\mathcal{L}^2[0,1]$. Let t possess both properties (the set of such points also has full measure). Then the previous equality means that

$$\lambda_n \overline{e_n(t)} = \bigl(\mathcal{K}(\,\cdot\,,t), e_n\bigr),$$

which gives the equality

$$\sum_{n=1}^{\infty} \lambda_n^2 |e_n(t)|^2 = \int_0^1 |\mathcal{K}(s,t)|^2\,ds. \tag{7.10.10}$$

After integration in t we obtain

$$\sum_{n=1}^{\infty} \lambda_n^2 = \int_0^1 \int_0^1 |\mathcal{K}(s,t)|^2\,ds\,dt < \infty.$$

Since for almost every t we have the orthogonal expansion

$$\mathcal{K}(\,\cdot\,,t) = \sum_{n=1}^{\infty} \bigl(\mathcal{K}(\,\cdot\,,t), e_n\bigr)e_n = \sum_{n=1}^{\infty} \lambda_n \overline{e_n(t)} e_n,$$

we have

$$\mathcal{K}(s,t) = \sum_{n=1}^{\infty} \lambda_n e_n(s)\overline{e_n(t)}, \tag{7.10.11}$$

where the series converges in $L^2([0,1]\times[0,1])$. If we impose some additional conditions on $\mathcal{K}$, then a stronger conclusion can be obtained.

7.10.42. Example. Suppose that there is a number C such that

$$\int_0^1 |\mathcal{K}(t,s)|^2\,ds \leqslant C \quad \text{almost everywhere on } [0,1].$$

Then $\sum_{n=1}^{\infty} |\lambda_n e_n(t)|^2 \leqslant C$ a.e., moreover, for every $x \in L^2[0,1]$ the series

$$T_{\mathcal{K}} x(t) = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n(t) \tag{7.10.12}$$

converges absolutely and uniformly on some set E of full measure.

PROOF. Since we have $|\mathcal{K}(t,s)| = |\mathcal{K}(s,t)|$, the first assertion is obvious from (7.10.10). For E we take the set of those points t for which (7.10.10) and the inequality from the hypotheses are fulfilled. Now the estimate

$$\begin{aligned} \sum_{n=m}^{\infty} |\lambda_n (x, e_n) e_n(t)| &\leqslant \Bigl(\sum_{n=m}^{\infty} |\lambda_n e_n(t)|^2\Bigr)^{1/2} \Bigl(\sum_{n=m}^{\infty} |(x, e_n)|^2\Bigr)^{1/2} \\ &\leqslant C^{1/2} \Bigl(\sum_{n=m}^{\infty} |(x, e_n)|^2\Bigr)^{1/2} \end{aligned}$$

proves the second assertion. □

7.10.43. Theorem. (MERCER'S THEOREM) *Suppose that $\mathcal{K}$ is a continuous function such that the operator $T_{\mathcal{K}}$ is nonnegative, i.e., $(T_{\mathcal{K}} x, x) \geqslant 0$. Then the functions e_n corresponding to eigenvalues $\lambda_n \neq 0$ have continuous modifications and the series* (7.10.11) *and* (7.10.12) *converge absolutely and uniformly. In addition,*

$$\operatorname{tr} T_{\mathcal{K}} = \int_0^1 \mathcal{K}(t,t)\, dt.$$

PROOF. For $\lambda_n \neq 0$ the continuous version of e_n is given by the formula

$$e_n(t) = \lambda_n^{-1} \int_0^1 \mathcal{K}(t,s) e_n(s)\, ds.$$

We observe that $\mathcal{K}(t,t) \geqslant 0$. Indeed, for a fixed $\tau \in [0,1)$ we take the functions $x_n = nI_{[\tau,\tau+1/n]} I_{[0,1]}$. By assumption

$$0 \leqslant (T_{\mathcal{K}} x_n, x_n) = n^2 \int_{\tau}^{\min(\tau+1/n,1)} \int_{\tau}^{\min(\tau+1/n,1)} \mathcal{K}(t,s)\, dt\, ds.$$

As $n \to \infty$, the integral in the right-hand side tends to $\mathcal{K}(\tau,\tau)$, hence $\mathcal{K}(\tau,\tau) \geqslant 0$. Let us apply this observation to the continuous kernels $\mathcal{K} - \mathcal{K}_n$, where

$$\mathcal{K}_n(t,s) = \sum_{j=1}^{n} \lambda_j e_j(t) \overline{e_j(s)},$$

defining nonnegative operators, since $\lambda_j \geqslant 0$ and

$$(T_{\mathcal{K}} x, x) - (T_{\mathcal{K}_n} x, x) = \sum_{j=n+1}^{\infty} \lambda_j |(x, e_j)|^2 \geqslant 0.$$

Thus, $\mathcal{K}_n(t,t) \leqslant \mathcal{K}(t,t)$. Therefore,

$$\sum_{j=1}^{n} \lambda_j |e_j(t)|^2 \leqslant \mathcal{K}(t,t) \leqslant M := \sup_{t,s} |\mathcal{K}(t,s)|.$$

Hence
$$\sum_{j=1}^{\infty}\lambda_j|e_j(t)e_j(s)|\leqslant\sum_{j=1}^{\infty}\lambda_j\frac{|e_j(t)|^2+|e_j(s)|^2}{2}\leqslant M.$$
In addition, for every fixed s the series $\sum_{j=1}^{\infty}\lambda_j e_j(t)\overline{e_j(s)}$ converges uniformly in t. Indeed, for any given $\varepsilon>0$ we can find m with $\sum_{j=m}^{\infty}\lambda_j|e_j(s)|^2\leqslant\varepsilon$ and by the Cauchy–Bunyakovskii inequality we obtain
$$\sum_{j=m}^{\infty}\lambda_j|e_j(t)e_j(s)|\leqslant\Bigl(\sum_{j=m}^{\infty}\lambda_j|e_j(t)|^2\Bigr)^{1/2}\Bigl(\sum_{j=m}^{\infty}\lambda_j|e_j(s)|^2\Bigr)^{1/2}\leqslant(M\varepsilon)^{1/2}.$$
Now we can conclude that the series $\sum_{n=1}^{\infty}\lambda_n e_n(t)\overline{e_n(s)}$ converges to $\mathcal{K}(t,s)$ at every point, not only almost everywhere. Indeed, let us denote the sum of this pointwise convergent series by $Q(t,s)$. It follows from what we have proved that the function Q is bounded and continuous in every variable separately. Let us fix s and show that $Q(t,s)=\mathcal{K}(t,s)$ for all t. By the continuity of both functions in t it suffices to verify that they have equal inner products with all functions e_k. Since the series defining Q converges uniformly in t (which is shown above), the integral of $Q(t,s)\overline{e_k(t)}$ in t equals $\lambda_k\overline{e_k(s)}$. The same value has the integral of $\mathcal{K}(t,s)\overline{e_k(t)}$ in t according to our choice of versions of e_k. The pointwise equality $\mathcal{K}(t,s)=Q(t,s)$ yields that the series $\sum_{n=1}^{\infty}\lambda_n|e_n(t)|^2$ converges pointwise to the continuous function $\mathcal{K}(t,t)$. By Dini's theorem this convergence is uniform. From this with the aid of the Cauchy–Bunyakovskii inequality one can easily derive the uniform convergence of series (7.10.11). The equality for the trace is obtained by integration of this series for $t=s$. □

For a generalization, see Exercise 7.10.128.

It is readily verified that for a continuous real symmetric kernel $\mathcal{K}$ on $[0,1]^2$ the corresponding operator $T_{\mathcal{K}}$ is nonnegative (in the sense of quadratic forms) precisely when the matrices $\bigl(\mathcal{K}(t_i,t_j)\bigr)_{i,j\leqslant n}$ are nonnegative definite. Hence the condition $T_{\mathcal{K}}\geqslant 0$ does not follow from the condition $\mathcal{K}(t,s)\geqslant 0$. As an example let us consider the kernel $\mathcal{K}(t,s)=|t-s|$ that generates an operator which is not nonnegative (of course, this can be easily seen from the formula for the trace in §7.10(iv)).

Note also that the nonnegativity of an operator T on $L^2(\mu)$ in the sense of quadratic forms should not be confused with the nonnegativity in the sense of ordered spaces, i.e., with the condition that $Tx\geqslant 0$ whenever $x\geqslant 0$ (none of these two conditions follows from the other).

7.10(vi). Tensor products

Let X and Y be Banach spaces. For every pair (x,y) in $X\times Y$, the formula $l\mapsto l(x)y$ defines a one-dimensional operator from X^* to Y; this operator will be denoted by the symbol $x\otimes y$. Denote by $X\otimes Y$ the linear space in $\mathcal{L}(X^*,Y)$ generated by all operators $x\otimes y$. Note that a representation of an operator in the form $x_1\otimes y_1+\cdots+x_n\otimes y_n$ is not unique. For example, $x\otimes(y+z)=x\otimes y+x\otimes z$. The linear space $X\otimes Y$ is called the *algebraic tensor product* of the spaces X and Y.

It can be completed with respect to any norm on the space of finite-dimensional operators. A norm p on $X\otimes Y$ is called a cross-norm if $p(x\otimes y) = \|x\|\,\|y\|$ for all $x \in X$, $y \in Y$. It turns out that among cross-norms there are two extreme ones that correspond to the usual operator norm on $\mathcal{L}(X^*, Y)$ and the nuclear norm. These two cross-norms are given by the equalities

$$\|u\|_\infty := \varepsilon(u) := \sup\Bigl\{\sum_i f(x_i)g(y_i)\colon\ u = \sum_i x_i\otimes y_i, \|f\|_{X^*} \leqslant 1, \|g\|_{Y^*} \leqslant 1\Bigr\},$$

$$\|u\|_{\mathcal{N}} := \pi(u) := \inf\Bigl\{\sum_i \|x_i\|\,\|y_i\|\colon\ u = \sum_i x_i\otimes y_i\Bigr\}.$$

They are called the injective and projective norms, respectively.

For every cross-norm p we have $\|u\|_\infty \leqslant p(u) \leqslant \|u\|_{\mathcal{N}}$ (Exercise 7.10.119). The term the "nuclear norm" is closely connected with the fact that an operator $T \in \mathcal{L}(X, Y)$ is called nuclear if it is representable in the form

$$Tx = \sum_{i=1}^\infty u_i(x)v_i, \quad \text{where } u_i \in X^*,\ v_i \in Y,\ \sum_{i=1}^\infty \|u_i\|\,\|v_i\| < \infty.$$

The infimum of sums $\sum_{i=1}^\infty \|u_i\|\,\|v_i\|$ over all possible representations of T is called the nuclear norm of T and is denoted by the symbol $\|T\|_{\mathcal{N}}$.

The completions of $X\otimes Y$ with respect to the norms $\|\cdot\|_{\mathcal{N}}$ and $\|\cdot\|_\infty$ are denoted by the symbols $X\widehat{\otimes}Y$ and $X\widetilde{\otimes}Y$, respectively. Alternative symbols are $X\hat{\otimes}_\pi Y$ and $X\hat{\otimes}_\varepsilon Y$.

The space $X\widehat{\otimes}Y = X\hat{\otimes}_\pi Y$ is called the *Banach tensor product* of X and Y. Every element $u \in X\widehat{\otimes}Y$ can be represented as a series

$$u = \sum_{i=1}^\infty x_i\otimes y_i, \quad \text{where } \sum_{i=1}^\infty \|x_i\|\,\|y_i\| < \infty,$$

and $\|u\|_{\mathcal{N}}$ is the infimum of sums of the indicated form (this is clear from Exercise 5.6.51).

If X and Y are Hilbert spaces, then $X\otimes Y$ can be equipped with the Hilbert–Schmidt norm, which leads to the *Hilbert tensor product* $X\otimes_2 Y$. Thus, in the case where $X = Y = H$ is a Hilbert space, the Banach tensor product $H\widehat{\otimes}H$ is the space $\mathcal{N}(H)$ of nuclear operators, the Hilbert tensor product $H\otimes_2 H$ is the space $\mathcal{H}(H)$ of Hilbert–Schmidt operators, and the tensor product $H\widetilde{\otimes}H$ is the space $\mathcal{K}(H)$ of compact operators, moreover, the tensor norms introduced above are exactly the corresponding operator norms.

If A is a bounded operator on X and B is a bounded operator on Y, then $A\otimes B$ is defined as a bounded operator on $X\otimes Y$ by setting

$$A\otimes B(u\otimes v) = Au\otimes Bv.$$

If p is a cross-norm on $X\otimes Y$, then for $z = \sum_i x_i\otimes y_i$ we have

$$p(A\otimes Bz) \leqslant \|A\|\,\|B\|\sum_i \|x_i\|\,\|y_i\|,$$

hence the left-hand side is estimated by $\|A\|\,\|B\|\,\|z\|_{\mathcal{N}}$. In particular, $A\otimes B$ is bounded on the Banach tensor product and its norm is $\|A\|\,\|B\|$. It is readily seen that the same is true for the injective norm. Cross-norms with this property are called uniform in Ryan [**523**, p. 128]. If $X = Y = H$ is a Hilbert space, then $A\otimes B$ is also bounded on the Hilbert tensor product and its norm is $\|A\|\,\|B\|$. Indeed, it suffices to prove that the norm is bounded by $\|A\|\,\|B\|$. Moreover, it suffices to do that in the case where one of the operators is the identity, since

$$A\otimes B = (A\otimes I)(I\otimes B).$$

Now let $B = I$. If $v = \sum_i u_i\otimes v_i$, where v_i are mutually orthogonal, then

$$\|v\|_{\mathcal{H}}^2 = \sum_i \|u_i\otimes v_i\|^2 = \sum_i \|u_i\|^2.$$

Hence

$$\|(A\otimes I)v\|_{\mathcal{H}}^2 = \Bigl\|\sum_i Au_i\otimes v_i\Bigr\|_{\mathcal{H}}^2 = \sum_i \|Au_i\|^2 \leqslant \|A\|^2\sum_i \|u_i\|^2 = \|A\|^2\|v\|_{\mathcal{H}}^2.$$

A useful application of this construction can be found in Exercise 7.10.135.

7.10(vii). Fredholm operators

Let X and Y be Banach spaces.

7.10.44. Definition. *A bounded operator $A\colon\ X \to Y$ is called Fredholm if its kernel* $\operatorname{Ker} A$ *has a finite dimension and its range* $\operatorname{Ran} A = A(X)$ *has a finite codimension.*

The number $\dim\operatorname{Ker} A - \operatorname{codim}\operatorname{Ran} A$ *is called the index of the operator A and denoted by the symbol* $\operatorname{Ind} A$.

According to Proposition 6.2.12 the range of any Fredholm operator is closed.

Sometimes Fredholm operators are called Noether operators, while the term Fredholm is reserved for Noether operators with zero index.

The index of a Fredholm operator turns out to be a more important characteristic than the dimension of its kernel or the codimension of its range. This is connected with the behavior of the index under compositions and perturbations of operators, which will be demonstrated below.

7.10.45. Example. (i) Every linear operator A on a finite-dimensional space E is Fredholm with zero index, since

$$\dim\operatorname{Ker} A + \dim A(E) = \dim E.$$

(ii) A compact operator $K\colon\ X \to Y$ can be Fredholm only in the case where both spaces X and Y are finite-dimensional. Indeed, the closed space $K(X)$ must be finite-dimensional by the open mapping theorem (the image of an open ball is open in $K(X)$, but in this case it is totally bounded). Then the kernel of K has a finite codimension, i.e., X is also finite-dimensional.

(iii) Let $K\colon\ X \to X$ be a compact operator. Then the operator $I + K$ is Fredholm. In addition, $\operatorname{Ind}(I+K) = 0$. This was proved in §7.4. As we shall see below, this example is very typical. The term a “Fredholm operator” gives credit to Fredholm, who investigated integral equations. His results concerned with integral operators paved the way to subsequent research of Riesz and Schauder about abstract compact operators. The Nikolskii theorem proved below exhibits a close connection between Fredholm operators and compact perturbations of the identity operator.

(iv) The operator $A_\varphi\colon L^2[0,1] \to L^2[0,1]$ of multiplication by a bounded measurable function φ is Fredholm precisely when it is invertible, i.e., $|\varphi(t)| \geqslant c$ almost everywhere on $[0,1]$ for some number $c > 0$. In this case the index of A_φ is zero. Indeed, the set $Z := \varphi^{-1}(0)$ has measure zero, since otherwise the space of functions from $L^2[0,1]$ vanishing outside Z is infinite-dimensional. This space coincides with the kernel of A. Thus, the operator A is injective. In addition, its range is everywhere dense in $L^2[0,1]$, because for every function f in $L^2[0,1]$ the functions fI_{E_n}, where $E_n := \{t\colon |\varphi|(t) \geqslant n^{-1}\}$, converge to f in $L^2[0,1]$, but such functions belong to the range of A. Hence the operator A is one-to-one.

(v) Let U be the open unit disc in $\mathbb{C}$ and let $X = H(U)$ be the Banach space of analytic functions on U continuous on the closure of U with the norm $\|f\| = \max_{z\in U} |f(z)|$. Set $Af(z) = z^n f(z)$. Then A is a Fredholm operator with index n. Indeed, this operator has the zero kernel and a range of finite codimension. The functions $1, z, \ldots, z^{n-1}$ along with the range of A generate $H(U)$, which is clear from the expansion $f(z) = \sum_{k=0}^\infty a_k z^k$.

Let us consider in greater detail the structure of a Fredholm operator T between two Banach spaces X and Y. Let us take a closed linear complement X_1 to the kernel of T in X (see Corollary 6.4.2). There is also a finite-dimensional complement Y_0 to the range Y_1 of the operator T. The operator T maps X_1 one-to-one to Y_1 and equals zero on $X_0 := \operatorname{Ker} T$. Since X_1 and Y_1 are Banach spaces, the operator $T\colon X_1 \to Y_1$ is a linear homeomorphism. This yields the following assertion.

7.10.46. Proposition. *Let X, Y, Z be Banach spaces and let $T \in \mathcal{L}(X,Y)$ be a Fredholm operator. Then the image $T(Z)$ of every closed linear subspace $Z \subset X$ is closed in Y.*

PROOF. The linear subspace $Z_1 := Z \cap X_1$ is closed and has a finite-dimensional linear complement Z_2 to Z. Hence $T(Z) = T(Z_1) + T(Z_2)$, where the subspace $T(Z_1)$ is closed by the property that T is a homeomorphism on X_1 and the subspace $T(Z_2)$ is finite-dimensional. According to Proposition 5.3.7 the set $T(Z)$ is closed. $\square$

Now with the aid of the above considerations we prove the following theorem due to S. M. Nikolskii, which connects Fredholm operators with compact perturbations of the identity.

7.10.47. Theorem. *Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. The operator T is Fredholm precisely when there exists an operator $S \in \mathcal{L}(Y,X)$ such that the operators $ST - I_X$ and $TS - I_Y$ are compact* (*on X and Y, respectively*). *In this case the operator S can be chosen such that the indicated operators will be even finite-dimensional.*

PROOF. Let T be Fredholm. As above, we take a closed linear complement X_1 to $X_0 = \operatorname{Ker} T$ in X. We know that the projection operator $P_1\colon X \to X_1$ is continuous. The projection $P = I_X - P_1$ onto $\mathrm{Ker}\, T$ is also continuous. There is a finite-dimensional complement Y_0 to the range Y_1 of the operator T and the continuous projection operator $Q_1\colon Y \to Y_1$. Then $Q := I_Y - Q_1$ is the continuous projection operator onto Y_0. The operator T maps X_1 one-to-one onto Y_1. The Banach inverse mapping theorem gives a mapping $S_0 \in \mathcal{L}(Y_1, X_1)$ with $S_0 T x = x$ for all $x \in X_1$ and $T S_0 y = y$ for all $y \in Y_1$. Set $S\colon Y \to X$, $Sy = S_0 Q_1 y$. Then $S \in \mathcal{L}(Y, X)$ and $TSy = TS_0 Q_1 y = Q_1 y = y - Qy$ for all vectors $y \in Y$. In addition, we have

$$STx = S_0 P_0 T x = S_0 T x = S_0 T P_0 x = P_0 x = x - Px \quad \text{for all } x \in X.$$

Thus, the operators $ST - I_X$ and $TS - I_Y$ are finite-dimensional.

Conversely, suppose that there exists an operator $S \in \mathcal{L}(Y, X)$ such that the operators $ST - I_X$ and $TS - I_Y$ are compact. This gives the Fredholm property of the operators ST and TS. Since $\mathrm{Ker}\, T \subset \mathrm{Ker}\, ST$, the kernel of T is finite-dimensional. In addition, the range of TS is contained in the range of T, hence the latter has a finite codimension. □

Note that the operator S ("almost inverse" to the Fredholm operator T) is also Fredholm, since it satisfies the Nikolskii condition.

7.10.48. Theorem. *Let X, Y and Z be Banach spaces and let $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(Y, Z)$ be Fredholm operators. Then the operator $TS \in \mathcal{L}(X, Z)$ is also Fredholm and* $\mathrm{Ind}\,(TS) = \mathrm{Ind}\, T + \mathrm{Ind}\, S$.

PROOF. Let us consider the finite-dimensional subspace $Y_0 := S(X) \cap \mathrm{Ker}\, T$ in Y. It is clear that $\mathrm{Ker}\,(TS) = S^{-1}(\mathrm{Ker}\, T)$. The operator S maps $\mathrm{Ker}\,(TS)$ onto Y_0 and the kernel of this mapping is the set $\mathrm{Ker}\, S$. Thus,

$$\dim \mathrm{Ker}\,(TS) = \dim \mathrm{Ker}\, S + \dim Y_0.$$

Since the range of S is finite-dimensional, the subspace $S(X) + \mathrm{Ker}\, T$ possesses a finite-dimensional algebraic complement Y_1. It is clear that $\mathrm{Ran}\, T$ is the algebraic sum of $\mathrm{Ran}\,(TS)$ and $T(Y_1)$. Let us verify that this is a direct sum. Indeed, if $TSx = Ty$, where $x \in X$ and $y \in Y_1$, then $Sx - y \in \mathrm{Ker}\, T$. Then we have $y \in \bigl(S(X) + \mathrm{Ker}\, T\bigr) \cap Y_1$. Hence $y = 0$, since the intersection above consists only of zero. Thus, the operator TS is Fredholm, and we arrive at the equality

$$\mathrm{codim}\, \mathrm{Ran}\,(TS) = \mathrm{codim}\, \mathrm{Ran}\, T + \dim Y_1.$$

On the ground of the already proved facts we obtain the following relation:

$$\mathrm{Ind}\,(TS) = (\dim \mathrm{Ker}\, S + \dim Y_0) - (\mathrm{codim}\, \mathrm{Ran}\, T + \dim Y_1).$$

The subspace Y_0 in the finite-dimensional subspace $\mathrm{Ker}\, T$ has a linear complement Y_2. Hence $\dim \mathrm{Ker}\, T = \dim Y_0 + \dim Y_2$. For obtaining the desired relation it remains to observe that

$$\mathrm{codim}\, \mathrm{Ran}\, S = \dim Y_1 + \dim Y_2.$$

This equality follows from the fact that by construction Y is the direct sum of $\mathrm{Ran}\, S + \mathrm{Ker}\, T$ and Y_1, moreover, $\mathrm{Ran}\, S + \mathrm{Ker}\, T$ is a direct sum of $\mathrm{Ran}\, S$ and Y_2 due to the fact that Y_2 is a complement of $\mathrm{Ran}\, S \cap \mathrm{Ker}\, T$ in $\mathrm{Ker}\, T$. □

7.10.49. Corollary. *Let X and Y be Banach spaces, let $T \in \mathcal{L}(X,Y)$ be a Fredholm operator, and let $K \in \mathcal{K}(X,Y)$ be a compact operator. Then the operator $T+K$ is also Fredholm, moreover,*

$$\operatorname{Ind}(T+K) = \operatorname{Ind} T.$$

PROOF. By the Nikolskii theorem there is an operator $S \in \mathcal{L}(Y,X)$ such that the operators $P_1 := ST - I_X$ and $P_2 := TS - I_Y$ are finite-dimensional. Hence $S(T+K) = I_X + P_1 + SK$ and $(T+K)S = I_Y + P_2 + KS$, where the operators $P_1 + SK$ and $P_2 + KS$ are compact. By the same theorem the operator $T+K$ is Fredholm. It has been noted in Example 7.10.45(iii) that compact perturbations of the identity operator have zero index. By the theorem proved above this gives the equality

$$\operatorname{Ind} T + \operatorname{Ind} S = 0 = \operatorname{Ind}(T+K) + \operatorname{Ind} S,$$

which yields the equality $\operatorname{Ind}(T+K) = \operatorname{Ind} T$. □

7.10.50. Theorem. *Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is Fredholm precisely when its adjoint operator T^* is Fredholm. Moreover,* $\operatorname{Ind} T = -\operatorname{Ind} T^*$.

PROOF. Let T be Fredholm. Then

$$\dim \operatorname{Ker} T^* = \operatorname{codim} \operatorname{Ran} T, \quad \operatorname{codim} \operatorname{Ran} T^* = \dim \operatorname{Ker} T.$$

Indeed, let us represent X as $X_0 \oplus X_1$, where $X_0 = \operatorname{Ker} T$ and X_1 is a closed linear complement to X_0. In addition, let us represent Y as $Y = Y_0 \oplus Y_1$, where Y_1 is the closed range of T and Y_0 is a finite-dimensional complement to Y_1. In this representation the operator T is written as $(x_0, x_1) \mapsto (0, T_1 x_1)$, where $T_1 \colon X_1 \to Y_1$ is an invertible operator. Then the operator $T^* \colon\ Y_0^* \oplus Y_1^* \to X_0^* \oplus X_1^*$ acts as follows: $T^*(y_0^*, y_1^*) = (0, T_1^* y_1^*)$, $y_0^* \in Y_0^*$, $y_1^* \in Y_1^*$. The operator T_1^* is an isomorphism between the spaces Y_1^* and X_1^*. For finite-dimensional spaces we have the equalities $\dim X_0 = \dim X_0^*$ and $\dim Y_0 = \dim Y_0^*$. This shows that T^* is Fredholm and gives the desired equality. Note that the Fredholm property of T^* by itself is obvious from the Nikolskii theorem and the equalities $(ST)^* = T^*S^*$, $(TS)^* = S^*T^*$. Conversely, let the operator $T^* \colon\ Y^* \to X^*$ be Fredholm. Then the operator $T^{**} \colon\ X^{**} \to Y^{**}$ is also Fredholm. Hence $\dim \operatorname{Ker} T < \infty$. The range of T has a finite codimension. Indeed, this range is closed, since it coincides with the image of the closed subspace X in X^{**} under the action of the Fredholm operator T^{**}. Therefore, the range of T coincides with the annihilator of the finite-dimensional kernel of T^* (see Lemma 6.8.1). □

It is suggested in Exercise 7.10.120 to prove that a Fredholm operator T has zero index precisely when $T = S + K$, where the operator S is invertible and the operator K is finite-dimensional.

In many infinite-dimensional Banach spaces one can easily construct operators that are neither compact nor Fredholm. For a long time the following problem remained open: does there exist an infinite-dimensional Banach space in which every bounded operator has the form $\lambda I + K$, where K is compact? Only recently S. Argyros and R. Haydon [**666**] have constructed such a space.

7.10(viii). The vector form of the spectral theorem

Here we obtain two more functional representations of selfadjoint operators, but we employ spaces of vector functions. The next assertion follows directly from the proof of Theorem 7.8.6.

7.10.51. Theorem. *Let A be a selfadjoint operator on a separable Hilbert space $H \neq 0$. Then there exists a finite or countable family of nonnegative Borel measures μ_n on $[-\|A\|, \|A\|]$ such that the operator A is unitarily equivalent to the operator B on the space $\bigoplus_{n=1}^{\infty} L^2(\mu_n)$ acting by the formula*

$$Bf(t) = \bigl(tf_1(t), tf_2(t), \ldots, tf_n(t), \ldots\bigr), \quad f = (f_1, f_2, \ldots, f_n, \ldots).$$

This theorem can be reformulated in a different form, where the operator acts not in the sum of the spaces $L^2(\mu_n)$, but in a subspace of some common space $L^2(\mu, H)$ of square integrable vector functions. Let us first introduce the space $L^2(\mu, H)$, where μ is a finite nonnegative measure on a σ-algebra $\mathcal{A}$ in a space Ω and H is a separable Hilbert space. A mapping x with values in H will be called μ-measurable if it is defined μ-a.e. and for some orthonormal basis $\{e_n\}$ in H the scalar functions $\omega \mapsto \bigl(x(\omega), e_n\bigr)$ are μ-measurable. This property does not depend on the choice of a basis: if $\{\varphi_k\}$ is another orthonormal basis, then

$$\bigl(x(\omega), \varphi_k\bigr) = \textstyle\sum_{n=1}^{\infty} \bigl(x(\omega), e_n\bigr)_H (e_n, \varphi_k)_H.$$

Similarly to the scalar case one introduces the class $\mathcal{L}^2(\mu, H)$ of all μ-measurable mappings x with values in H such that $\|\cdot\|_H^2 \in L^1(\mu)$, i.e.,

$$\sum_{n=1}^{\infty} \int_{\Omega} \bigl|\bigl(x(\omega), e_n\bigr)_H\bigr|^2 \, \mu(d\omega) < \infty.$$

Finally, let $L^2(\mu, H)$ denote the space of equivalence classes in $\mathcal{L}^2(\mu, H)$, where equivalent mappings are μ-a.e. equal mappings (see §6.10(vi)). As in the scalar case, the space $L^2(\mu, H)$ is equipped with a structure of a linear space by means of operations on representatives of classes and the inner product is defined by

$$(x, y) := \int_{\Omega} \bigl(x(\omega), y(\omega)\bigr)_H \, \mu(d\omega) = \sum_{n=1}^{\infty} \int_{\Omega} \bigl(x(\omega), e_n\bigr)_H \overline{\bigl(y(\omega), e_n\bigr)_H} \, \mu(d\omega),$$

where in the right-hand side one takes representatives of equivalence classes. In particular, the mapping x has the norm

$$\|x\| := \biggl(\int_{\Omega} \|x(\omega)\|_H^2 \, \mu(d\omega)\biggr)^{1/2}.$$

Exercise 7.10.98 suggests to verify the completeness of $L^2(\mu, H)$. Choosing orthonormal bases $\{e_n\}$ in H and $\{f_k\}$ in $L^2(\mu)$, we obtain an orthonormal basis in $L^2(\mu, H)$ consisting of all mappings $\omega \mapsto f_k(\omega)e_n$.

7.10.52. Theorem. *Let A be a selfadjoint operator on a separable Hilbert space $H \neq 0$. Then there exists a nonnegative Borel measure μ on $[-\|A\|, \|A\|]$ such that the operator A is unitarily equivalent to the operator A_0 defined by*

$$A_0 f(t) = tf(t)$$

on some closed linear subspace of the Hilbert space $L^2(\mu, l^2)$.

PROOF. Let $\|A\| \leqslant 1$. We know that the operator A is unitarily equivalent to the direct sum of operators of multiplication by the argument on the spaces $L^2(\mu_n)$ with some nonnegative Borel measures μ_n on $\Omega := \sigma(A)$, where $\mu_n(\Omega) \leqslant 1$. For μ we take the measure $\mu := \sum_{n=1}^{\infty} 2^{-n}\mu_n$ on Ω. Hence it suffices to prove our assertion for this direct sum. Since every measure μ_n is obviously absolutely continuous with respect to the measure μ, by the Radon–Nikodym theorem it possesses a density ϱ_n with respect to μ, i.e., there exists a nonnegative μ-integrable function ϱ_n such that

$$\mu_n(B) = \int_B \varrho_n(t)\,\mu(dt)$$

for every Borel set $B \subset \Omega$. This equality yields (see §3.9) that for every μ_n-integrable function φ we have

$$\int_\Omega \varphi\, d\mu_n = \int_\Omega \varphi\varrho_n\, d\mu.$$

In particular, this is true for all $\varphi \in L^2(\mu_n)$. Let us define an embedding J of the space $\bigoplus_{n=1}^{\infty} L^2(\mu_n)$ into $L^2(\mu, l^2)$ by the formula

$$J\bigl(\{\varphi_n\}_{n=1}^{\infty}\bigr)(t) = \{\varphi_n(t)\sqrt{\varrho_n}(t)\}_{n=1}^{\infty}.$$

It is clear that J is linear and preserves the inner product, since for every function $\varphi = \{\varphi_n\}_{n=1}^{\infty}$ in $\bigoplus_{n=1}^{\infty} L^2(\mu_n)$ we have

$$\|J\varphi\|^2 = \sum_{n=1}^{\infty}\int_\Omega |\varphi_n(t)|^2\varrho_n(t)\,\mu(dt) = \sum_{n=1}^{\infty}\int_\Omega |\varphi_n(t)|^2\,\mu_n(dt) = \|\varphi\|^2.$$

Therefore, the range of the isometry J is a closed linear subspace E in $L^2(\mu, l^2)$. In the general case this subspace does not coincide with $L^2(\mu, l^2)$. However, the operator of multiplication by the argument acts on E by the natural formula $A_0(\{x_n\})(t) = \{tx_n(t)\}$ and corresponds to our operator on the space $\bigoplus_{n=1}^{\infty} L^2(\mu_n)$ under the isomorphism J. □

7.10(ix). Invariant subspaces

In our study of selfadjoint operators we have occasionally made use of their invariant subspaces, i.e., closed subspaces H_0 such that $A(H_0) \subset H_0$. Every selfadjoint operator has many such subspaces (Exercise 7.10.91). However, already for several decades the following question remains open: does every bounded operator on a separable complex Hilbert space H have invariant closed subspaces different from 0 and H? The same question for general separable Banach spaces also remained open for a long time, but in 1981 P. Enflo constructed a counter-example (later C. Read constructed a counter-example in l^1). Here we present a remarkable result of V. I. Lomonosov obtained before the discovery of these counter-examples and strengthening some results due to von Neumann, Aronszajn and Smith. The proof gives an unexpected and beautiful application of the nonlinear Schauder theorem to linear operators.

7.10.53. Theorem. *Let $K \neq 0$ be a compact linear operator on an infinite-dimensional Banach space X. Then all bounded operators commuting with K (including K itself) have a common nontrivial closed invariant subspace.*

PROOF. Suppose the contrary and take an open ball $U \subset X$ such that 0 does not belong to the convex compact set $S := \overline{K(U)}$. Let us consider the subalgebra $\mathcal{M} := \{A \in \mathcal{L}(X)\colon AK = KA\}$ and the linear subspace $M(x) := \{Ax\colon A \in \mathcal{M}\}$ for every $x \in X$. For $x \neq 0$ we have $\overline{M(x)} = X$, since the closed subspace $\overline{M(x)}$ is invariant with respect to all $A \in \mathcal{M}$ for all $x \in M(x)$, because $I \in \mathcal{M}$. Hence for every $s \in S$ there exists $A_s \in \mathcal{M}$ with $A_s(s) \in U$. This gives an open ball V_s centered at s with $A_s(V_s) \subset U$. By the compactness of S we obtain a finite collection $s_1, \ldots, s_n \in S$ such that $S \subset V_{s_1} \cup \ldots \cup V_{s_n}$. As shown in §1.9(iv), there exist functions $\varphi_i \in C(S)$, $i = 1, \ldots, n$, such that $\varphi_i \geqslant 0$, $\sum_{i=1}^n \varphi_i(x) = 1$ and $\varphi_i(x) = 0$ whenever $x \notin V_{s_i}$. Let us consider the mapping

$$F\colon S \to X, \quad F(x) = K\Bigl(\sum_{i=1}^n \varphi_i(x) A_{s_i}(x)\Bigr).$$

This mapping is continuous and $F(S) \subset S$, since for any $x \in S$ we have $\varphi_i(x) \neq 0$ only for $x \in V_{s_i}$, but then $A_{s_i}(x) \in U$ and the set U is convex. By Schauder's theorem there is a point $x_0 \in S$ with $F(x_0) = x_0$. Finally, let us take the compact operator $T = \sum_{i=1}^n \varphi_i(x_0) K A_{s_i}$. It is clear that $T \in \mathcal{M}$ and $T(x_0) = x_0$. The closed subspace $L = \mathrm{Ker}\,(T - I)$ is finite-dimensional, $L \neq 0$. Since $KT = TK$, we have $K(L) \subset L$. Hence in L the operator K has an eigenvector with some eigenvalue λ. It remains to observe that the subspace $E = \mathrm{Ker}\,(K - \lambda I)$ is invariant with respect to all operators commuting with K. □

If X is a real space, the operator K itself has an invariant subspace. A similar reasoning gives the following result due to V. I. Lomonosov.

7.10.54. Theorem. *Let X be an infinite-dimensional Banach space. Suppose that an operator $T \in \mathcal{L}(X)$ commutes with some nonzero compact operator and is not a multiple of the unit operator. Then all bounded operators commuting with T have a common nontrivial closed invariant subspace.*

The class of operators T covered by this theorem is so large that it took several years to construct an operator that does not belong to it. About invariant subspaces, see [**52**] and [**662**].

Exercises

7.10.55.° Find the spectra of the following operators on l^2: (i) $Ax = (0, x_1, x_2, \ldots)$; (ii) $Ax = (x_2, x_3, \ldots)$; (iii) $Ax = (x_2/2, x_3/3, \ldots, x_n/n, \ldots)$.

HINT: use that the operators in (i) and (ii) are adjoint to each other and that the operator in (iii) is compact.

7.10.56.° Find the spectra of the following operators on the space $l^2(\mathbb{Z})$ of two-sided sequences: (i) $(Ax)_n = x_{n+1}$; (ii) $(Ax)_n = 0$ if n is odd, $(Ax)_n = x_{n+1}$ if n is even; (iii) $(Ax)_n = x_{n+1}/(|n| + 1)$.

7.10.57.° Let $\{e_n\}$ be an orthonormal basis in a separable Hilbert space H and let a bounded operator A be given by the formula $Ae_n = \alpha_n e_n$, where $\{\alpha_n\}$ is a bounded sequence in $\mathbb{C}$. Prove that the spectrum of A is the closure of $\{\alpha_n\}$.

7.10.58.° The operator A on l^2 is defined by $Ax = (r_1x_1, r_2x_2, \ldots, r_nx_n, \ldots)$, where $\{r_n\}$ is some enumeration of the set of rational numbers in $[0,1]$. Prove that A has a cyclic vector and the spectrum $[0,1]$, but is not unitarily equivalent to the operator of multiplication by the argument on $L^2[0,1]$.

7.10.59.° Investigate whether there are unitarily equivalent operators among the following operators of multiplication by a function φ on $L^2[a,b]$, where $[a,b]$ is equipped with Lebesgue measure: (a) $\varphi(t) = t$, $[a,b] = [0,1]$, (b) $\varphi(t) = |t|$, $[a,b] = [-1,1]$, (c) $\varphi(t) = t^2$, $[a,b] = [0,1]$, (d) $\varphi(t) = t^3$, $[a,b] = [0,1]$, (e) $\varphi(t) = t^{1/2}$, $[a,b] = [0,1]$, (f) $\varphi(t) = \sin t$, $[a,b] = [0,1]$.

7.10.60. (i) Prove that a selfadjoint operator on $\mathbb{C}^n$ possesses a cyclic vector precisely when it has no multiple eigenvalues. (ii) Prove that a compact selfadjoint operator on a Hilbert space possesses a cyclic vector precisely when it has no multiple eigenvalues.

HINT: write the operator in the diagonal form and observe that in case of an eigenvalue of multiplicity at least two for every vector h there is a nonzero vector u orthogonal to the orbit of h.

7.10.61. Suppose that μ is a Borel probability measure on $[0,1]$ mutually singular with Lebesgue measure λ, having no points of positive measure and satisfying the condition $\mu\big((a,b)\big) > 0$ whenever $0 \leqslant a < b \leqslant 1$. Prove that the operators of multiplication by the argument on $L^2(\mu)$ and $L^2(\lambda)$ have cyclic vectors, equal spectra and have no eigenvalues, but are not unitarily equivalent.

7.10.62.° Let A be a bounded operator on a complex Banach space X such that one has $\|A\| \in \sigma(A)$. Prove that $\|I + A\| = 1 + \|A\|$.

HINT: use that otherwise the norm of the operator $(1 + \|A\|)^{-1}(I + A)$ is less than 1, hence the difference with the unit operator must be invertible.

7.10.63.° Let H be a Hilbert space, $u, v \in H$, and let the operator $K_{u,v}$ be given by the formula $K_{u,v}x := (x,u)v$. Show that $K_{u,v}^* = K_{v,u}$ and $|K_{u,v}|x = \|v\|(x,u)u$.

7.10.64.° Let U be the shift in the space l^2 of two-sided sequences $(z_n)_{n\in\mathbb{Z}}$ defined by $Ue_n = e_{n+1}$. Set $Kz := (z, e_{-1})e_0$. Prove that the spectrum of $U - K$ coincides with the unit disc.

7.10.65.° Let A and B be linear operators on a linear space X. Show that the operators AB and BA have the same nonzero eigenvalues.

7.10.66.° Let X be a complex Banach space and $A \in \mathcal{L}(X)$. Prove that if A^2 has an eigenvalue, then A also does.

HINT: use that $A^2 - \lambda = (A + \lambda^{1/2})(A - \lambda^{1/2})$.

7.10.67.° Construct a bounded linear operator on the complex space l^2 such that its spectrum consists of the two points 0 and 1 that are not its eigenvalues.

HINT: consider first a compact operator without eigenvalues.

7.10.68.° Find the eigenvalues of the operator V^*V for the Volterra operator on $L^2[0,1]$ defined by $Vx(t) = \displaystyle\int_0^t x(s)\,ds$.

7.10.69. Let X be a Banach space and let $t \mapsto A_t$ be a mapping from $[0,1]$ to $\mathcal{L}(X)$ continuous in the operator norm such that there exists a number $C > 0$ for which $\|x\| \leqslant C\|A_t x\|$ for all $x \in X$ and $t \in [0,1]$. Prove that the invertibility of A_0 is equivalent to the invertibility of A_1. In particular, the operators $A_0, A_1 \in \mathcal{L}(X)$ are simultaneously invertible or non-invertible if the condition is fulfilled for $A_t := tA_1 + (1-t)A_0$, $t \in [0,1]$.

HINT: observe that if A_t is invertible for some t, then $\|A_t^{-1}\| \leqslant C$, hence its perturbations by operators with norm less that C^{-1} are invertible.

7.10.70. Let X be a complex Banach space and let $A \in \mathcal{L}(X)$. Suppose that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is a function analytic in the disc of radius $r > \|A\|$ centered at zero. Prove that $\sigma\bigl(f(A)\bigr) = f\bigl(\sigma(A)\bigr)$, where $f(A) := \sum_{n=0}^{\infty} c_n A^n$.

HINT: observe that $f(A) - f(\lambda)I = (A - \lambda I)g(A) = g(A)(A - \lambda I)$ is not invertible if $\lambda \in \sigma(A)$. Conversely, if $f(A) - \mu I$ is not invertible, then on a compact disc of some radius larger than $\|A\|$ we have $f(z) - \mu = (z - \mu_1)\cdots(z - \mu_k)h(z)$, where h has no zeros on this disc. Hence $h(A)$ is invertible, which means that some μ_i belongs to $\sigma(A)$, so $\mu = f(\mu_i) \in f\bigl(\sigma(A)\bigr)$.

7.10.71°. Let A be a selfadjoint operator and $A \geqslant 0$. Prove that $\|Ax\|^2 \leqslant \|A\|(Ax,x)$.

HINT: use that $A = A^{1/2}A^{1/2}$.

7.10.72°. Let A, B be selfadjoint operators such that $A, B \geqslant 0$ and $AB = BA$. Prove that $AB \geqslant 0$. Give an example showing that this is not always true if A and B do not commute.

HINT: observe that $\sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$, so $(ABx,x) = \|\sqrt{A}\sqrt{B}x\|^2$.

7.10.73°. Let B be a selfadjoint operator, $B \geqslant 0$ and $A = B^2$. Prove that $B = \sqrt{A}$.

HINT: write B as the multiplication by a function.

7.10.74°. Let A be a selfadjoint operator on a nonzero separable Hilbert space such that $\sigma(A) = K_1 \cup K_2$, where K_1 and K_2 are compact and $K_1 \cap K_2 = \varnothing$. Prove that A can be written as the direct sum of operators A_1 and A_2 with $\sigma(A_1) = K_1$ and $\sigma(A_2) = K_2$.

HINT: represent A as an operator of multiplication.

7.10.75°. Let P_1 and P_2 be the orthogonal projections on subspaces H_1 and H_2. Prove that P_1P_2 is the orthogonal projection if and only if $P_1P_2 = P_2P_1$, and in this case P_1P_2 is the projection onto $H_1 \cap H_2$.

7.10.76°. Let P_j, $j \in \mathbb{N}$, be orthogonal projections in a complex Hilbert space.

(i) Let P be an orthogonal projection such that $(P_j x, x) \to (Px, x)$ for all x. Prove that $\|P_j x - Px\| \to 0$ for all x, using that $(P_j x, y) \to (Px, y)$ for all x, y. (ii) Suppose that for every x the sequence $(P_j x, x)$ is increasing (or, for every x, is decreasing). Prove that there exists an orthogonal projection P for which $\|P_j x - Px\| \to 0$ for all x. (iii) Show that if P is an operator such that $\|P_j x - Px\| \to 0$ for all x, then P is a projection. (iv) Give an example of an operator P that is not a projection, but for which $(P_j x, y) \to (Px, y)$ for all x, y. For this consider the indicator functions of the sets $B_n \subset [0,1]$ such that B_n consists of the left halves of the intervals obtained by partitioning $[0,1]$ into 2^n equal pieces.

7.10.77°. Let H be a separable Hilbert space and P a projection-valued measure on the Borel σ-algebra of the real line with values in $\mathcal{P}(H)$. (i) Prove that there exists a finite nonnegative Borel measure μ on $\mathbb{R}$ such that all measures $\mu_x(B) = \bigl(P(B)x, x\bigr)$ are absolutely continuous with respect to μ. (ii) Prove that for every Borel set $B \subset \mathbb{R}^1$ one can find a sequence of sets B_j written as finite unions of intervals such that $P(B_j)x \to P(B)x$ for every $x \in H$.

7.10.78. Let H be a separable Hilbert space, $A \in \mathcal{L}(H)$ a selfadjoint operator, and ψ a bounded Borel function. Prove that there exist polynomials p_n such that the operators $p_n(A)$ converge to $\psi(A)$ on every vector.

HINT: let $A = A_f$ be the operator of multiplication on $L^2(\mu)$ by a bounded function f; the measure $\nu := \mu \circ f^{-1}$ is concentrated on a compact interval, hence one can find a uniformly bounded sequence of polynomials p_n that converges to ψ ν-a.e.; then the functions $p_n \circ f$ are uniformly bounded and μ-a.e. converge to $\psi \circ f$, which by the dominated convergence theorem gives convergence of $(p_n \circ f)x$ to $(\psi \circ f)x$ in $L^2(\mu)$ for every element $x \in L^2(\mu)$. One can also use Theorem 7.9.6 and the measure μ from (i) in the previous exercise.

7.10.79. Let A_n be selfadjoint operators on a Hilbert space H such that

$$(A_1x, x) \leqslant (A_2x, x) \leqslant \cdots \leqslant (A_nx, x) \leqslant \cdots$$

and $\sup_n \|A_n\| < \infty$. Prove that there exists a selfadjoint operator A on H for which $Ax = \lim\limits_{n\to\infty} A_nx$ for all $x \in H$.

HINT: define A through its quadratic form and apply Exercise 7.10.71 to $A - A_n$.

7.10.80. Let H be a Hilbert space and $A \in \mathcal{L}(H)$ a selfadjoint operator. Prove that there exist unique selfadjoint operators $A^+, A^- \geqslant 0$ such that $A^+A^- = A^-A^+ = 0$ and $A = A^+ - A^-$, and if $A_1, A_2 \in \mathcal{L}(H)$ are selfadjoint operators for which $0 \leqslant A_1 \leqslant A^+$, $0 \leqslant A_2 \leqslant A^-$ and $A = A_1 - A_2$, then $A_1 = A^+$ and $A_2 = A^-$.

HINT: take $A^+ = f_1(A)$, $A^- = f_2(A)$, $f_1(t) = \max(t, 0)$, $f_2(t) = -\min(t, 0)$.

7.10.81. Construct an example of a continuous quadratic form on a Banach space which cannot be decomposed into the difference of two nonnegative continuous quadratic forms.

HINT: construct a sequence of two-dimensional Banach spaces $(X_n, \|\cdot\|_n)$ with quadratic forms Q_n such that $|Q_n| \leqslant 1$ on the unit ball U_n in X_n, but the positive part of Q_n assumes the value 2^n on U_n; consider the space of bounded sequences $x = (x_n)$, where $x_n \in X_n$, with the norm $\|x\| = \sup\limits_n \|x_n\|_n$ (or its separable subspace of a sequences (x_n) with $\|x_n\|_n \to 0$), and the form $\sum_{n=1}^{\infty} n^{-2}Q_n$.

7.10.82. Prove that for every two given Hilbert spaces at least one is linearly isometric to a closed subspace of the other.

HINT: take orthonormal bases and compare their cardinalities.

7.10.83. Let H be a Hilbert space and let $A \in \mathcal{L}(H)$ have a dense range. Prove that the operator A is invertible precisely when $|A|$ is invertible. Give an example showing that this can be false if the range of A is not dense.

HINT: use the polar decomposition; consider the operator $x \mapsto (0, x_1, x_2, \ldots)$ on l^2.

7.10.84. (i) Let A be the operator on $C[0,1]$ defined by the formula

$$Ax(t) = \frac{1}{t}\int_0^t x(s)\,ds, \quad Ax(0) = x(0).$$

Prove that for the spectral radius we have $r(A) = 1$.

(ii) Find eigenvalues of the operator defined by the formula above on the spaces $C[0,1]$ and $L^2[0,1]$ (see Exercise 6.10.138). Prove that in both cases it is not compact.

7.10.85. Let X be a Banach space, $A \in \mathcal{L}(X)$ and λ a boundary point of the spectrum of A. Prove that there exists a sequence of vectors x_n in the space X such that $\|x_n\| = 1$ and $\|Ax_n - \lambda x_n\| \to 0$.

HINT: if $\|Ax-\lambda x\| \geqslant \varepsilon > 0$ whenever $\|x\|=1$, then $\|Ax-\lambda_n x\| \geqslant \varepsilon/2$ on the unit sphere for a sequence of points $\lambda_n \to \lambda$ from the resolvent set, so $\|(A-\lambda_n I)^{-1}\| \leqslant 2/\varepsilon$, hence $A-\lambda I$ must be invertible.

7.10.86. Let H be a Hilbert space, $A \in \mathcal{L}(H)$, and let $A = U|A|$ be the polar decomposition of A. Suppose that $A = WB$ and $B = W^*A$, where $B \in \mathcal{L}(H)$ is a nonnegative selfadjoint operator and $W \in \mathcal{L}(H)$ is an isometry on the closure of $B(H)$. Prove that $B = |A|$ and $W = U$ on the closure of $|A|(H)$.

7.10.87. Let μ be a bounded Borel measure on the real line and φ a bounded μ-measurable function. Prove that the operator A_φ of multiplication by φ on $L^2(\mu)$ is compact precisely when the restriction of μ to the set $\{t\colon\ \varphi(t) \neq 0\}$ is concentrated on a finite or countable set of points α_n and $\varphi(\alpha_n) \to 0$ if the set of these points is infinite.

HINT: use that the operator of multiplication by a function separated from zero is invertible and that in case of an atomless measure the L^2-space over a set of positive measure is infinite-dimensional.

7.10.88. Let μ be a bounded Borel measure on the real line and φ a bounded μ-measurable function. When does the operator A_φ of multiplication by φ on $L^2(\mu)$ have a closed range?

7.10.89. Construct an infinite measure μ and a bounded μ-measurable function φ for which the spectrum of the operator of multiplication by φ on $L^2(\mu)$ does not coincide with the set of essential values of φ.

7.10.90. Let A be a selfadjoint operator on a separable Hilbert space and Π_0 the corresponding resolution of the identity. Prove that $\sigma(A)$ coincides with the complement to the union of all intervals on which Π_0 is constant and also with the set of points λ such that $\Pi_0(\lambda) \neq \lim_{\lambda_n \downarrow \lambda} \Pi_0(\lambda_n)$.

HINT: it suffices to consider the case where A is the multiplication by the argument on the space $L^2(\mu)$ for some measure μ with support $\sigma(A)$ in some interval.

7.10.91.° Let A be a selfadjoint operator on an infinite-dimensional Hilbert space. Prove that A has nontrivial closed invariant subspaces.

HINT: consider the closed linear span of the orbit of a nonzero vector.

7.10.92. Consider the shift operator $Ux = (0, x_1, x_2, \ldots)$ on l^2. Prove that there is no compact operator $K \neq 0$ such that $UK = KU$.

HINT: use that $U^n K = KU^n$ and that the sequence $U^n x$ converges to zero weakly, hence $\|KU^n x\| \to 0$.

7.10.93. Prove that a selfadjoint operator $A \geqslant 0$ on a Hilbert space is compact precisely when for some $\alpha > 0$ (then for every $\alpha > 0$) the operator A^α is compact.

7.10.94. Let A be a selfadjoint operator on a Hilbert space H such that $A \geqslant 0$ and the range $A(H)$ is closed. Prove that the range $A^\alpha(H)$ is closed for all $\alpha > 0$. In case $\alpha \in \mathbb{N}$ prove the same without the assumption that A is nonnegative.

HINT: here A is a linear isomorphism of $A(H)$.

7.10.95. Let A be a selfadjoint operator with $A > 0$. Show that for any vector y the function $F(x) = (Ax, x) - 2\mathrm{Re}\,(x, y)$ attains its minimum $-(A^{-1}y, y)$ at $x_0 = A^{-1}y$.

HINT: use that $F(x) = \bigl(A(x - x_0), x - x_0\bigr) - (Ax_0, x_0)$.

7.10.96. Let A and B be selfadjoint operators on a Hilbert space H and $A \leqslant B$.

(i) Prove that $TAT^* \leqslant TBT^*$ for every $T \in \mathcal{L}(H)$.

(ii) Prove that if the operator A is invertible and $A \geqslant 0$, then the operator B is invertible and $B^{-1} \leqslant A^{-1}$.

(iii) Let $A \geqslant 0$. Show that for all $\alpha \in (0,1)$ one has

$$A^{\alpha} = c_{\alpha} \int_0^{+\infty} t^{\alpha-1} A(A+tI)^{-1}\, dt.$$

(iv) Prove that if $A \geqslant 0$, then $A^{\alpha} \leqslant B^{\alpha}$ for all $\alpha \in (0,1]$. Give an example where it is not true that $A^2 \leqslant B^2$.

HINT: to prove (ii) use the previous exercise. Check (iii) for operators of multiplication. Deduce (iv) from (iii) and (ii).

7.10.97. Prove that Theorem 7.2.3 and Corollary 7.2.4 are true for normal operators.

7.10.98. Prove the completeness of the space $L^2(\mu, H)$ introduced before Theorem 7.10.52. Prove the more general assertion from Theorem 6.10.68.

7.10.99. Let L be a linear subspace in a Hilbert space containing no infinite-dimensional closed subspaces. Prove that every operator $A \in \mathcal{L}(H)$ with $A(H) \subset L$ is compact.

HINT: reduce the assertion to the case of a separable space and a selfadjoint operator, then write A as a multiplication operator.

7.10.100. Let H be a Hilbert space and $A \in \mathcal{L}(H)$. Prove that a vector y belongs to the range $A(H)$ precisely when $\sup_x |(x,y)|/\|A^*x\| < \infty$, where $0/0 := 1$.

HINT: apply Theorem 7.10.15.

7.10.101. Let A_n and A be selfadjoint operators on a separable Hilbert space H and $f \in C_b(\mathbb{R}^1)$. (i) Suppose that $A_n \to A$ in the operator norm. Prove that $f(A_n) \to f(A)$ in the operator norm. In particular, if A_n and A are nonnegative, then $\sqrt{A_n} \to \sqrt{A}$ in the operator norm.

(ii) Suppose that $A_n x \to Ax$ for all $x \in H$. Prove that $f(A_n)x \to f(A)x$ for all vectors $x \in H$. In particular, if A_n and A are nonnegative, then $\sqrt{A_n} \to \sqrt{A}$ in the strong operator topology.

HINT: observe that in both cases $\{A_n\}$ is norm bounded and verify the assertions for polynomials; in (ii) also observe that $A_n x_n \to Ax$ if $x_n \to x$.

7.10.102. Let $A \in \mathcal{L}(H)$, where H is a Hilbert space. Show that A is a Hilbert–Schmidt operator precisely when there is a number C such that $\sum_{i=1}^n \|Ae_i\|^2 \leqslant C$ for every finite orthonormal collection $e_1, \ldots, e_n$. Moreover, $\|A\|_{\mathcal{H}}^2 \leqslant C$.

7.10.103. A Hilbert–Schmidt ellipsoid in a separable Hilbert space is the image of the closed unit ball with respect to a Hilbert–Schmidt operator.

(i) Prove that a set V is a Hilbert–Schmidt ellipsoid precisely when one can find an orthonormal sequence $\{\varphi_n\}$ and numbers $\alpha_n > 0$ with $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ such that

$$V = \Bigl\{x\colon\ x = \textstyle\sum_{n=1}^{\infty} x_n \varphi_n,\ \sum_{n=1}^{\infty} |x_n/\alpha_n|^2 \leqslant 1\Bigr\}.$$

(ii) (V. N. Sudakov) Prove that if a bounded set W is contained in no Hilbert–Schmidt ellipsoid, then there exists an orthonormal sequence $\{\varphi_n\}$ such that

$$\textstyle\sum_{n=1}^{\infty} \sup_{w \in W} |(w, \varphi_n)|^2 = \infty.$$

7.10.104. Let H be a Hilbert space and $A, B \in \mathcal{L}(H)$.

(i) Prove that $A(H) + B(H) = \sqrt{AA^* + BB^*}(H)$. In particular, if the operators A and B are selfadjoint and nonnegative, then $\sqrt{A+B}(H) = \sqrt{A}(H) + \sqrt{B}(H)$.

(ii) Let the operators A and B be selfadjoint and nonnegative and have closed ranges. Prove that the range of $A+B$ is closed precisely when the linear subspace $A(H)+B(H)$ is closed.

HINT: Apply Theorem 7.10.15.

7.10.105.* Let H be a Hilbert space and let $A, B \in \mathcal{L}(H)$ be two operators such that $H = A(H) + B(H)$. (i) Prove that there exist closed subspaces $H_1 \subset A(H)$ and $H_2 \subset B(H)$ such that $H_1 + H_2 = H$, $H_1 \cap H_2 = 0$ and $H_1^\perp \cap H_2^\perp = 0$.

(ii) Let $A(H)$ and $B(H)$ be dense. Prove that the set $A(H) \cap B(H)$ is also dense.

HINT: see [**682**], [**683**].

7.10.106. Let K be a compact operator on a separable complex Hilbert space H and let $\{\lambda_n(K)\}$ be all nonzero eigenvalues of K written in the order of decreasing of their absolute values taking into account their multiplicities. (i) Prove that there is an orthonormal system $\{e_n\}$ such that $(Ke_n, e_n) = \lambda_n(K)$ for all n.

(ii) Let $\{s_n(K)\}$ be all eigenvalues of $|K|$. Prove that for every $p \in [1, \infty)$ Weyl's inequality $\sum_{n=1}^\infty |\lambda_n(K)|^p \leqslant \sum_{n=1}^\infty |s_n(K)|^p$ holds. Note that the inequality is also true for summing from 1 to any finite N (see [**554**, §1.6]).

7.10.107. Let H be a complex Hilbert space and $A \in \mathcal{L}(H)$. Prove that the operator A is normal (see §7.10(ii)) precisely when $\|Ax\| = \|A^*x\|$ for all $x \in H$.

HINT: observe that $(AA^*x - A^*Ax, x) = 0$.

7.10.108. Let T be a bounded operator on a Hilbert space H. Prove that it can be represented in the form $T = UA$, where U is a unitary operator and $A \geqslant 0$ is a selfadjoint operator, precisely when the subspaces $\operatorname{Ker} T$ and $\operatorname{Ker} T^*$ are isometric.

HINT: observe that $\operatorname{Ker} T = |T|(H)^\perp$ and $\operatorname{Ker} T^* = T(H)^\perp$.

7.10.109. Prove that a bounded operator T on a Hilbert space H is normal precisely when it is representable in the form $T = UA$, where U is a unitary operator, A is a nonnegative selfadjoint operator and $UA = AU$.

HINT: show that T admits a polar decomposition with a unitary operator.

7.10.110. Suppose that a bounded operator T on a Hilbert space H has the form $T = UA$, where U is unitary, A is selfadjoint , $A \geqslant 0$, and $UA \neq AU$. Prove that T is not normal.

7.10.111. Show that any bounded linear operator $A\colon\ L^2[0,1] \to L^2[0,1]$ has the form

$$Ax(t) = \frac{d}{dt}\int_0^1 \mathcal{K}(t,s)x(s)\,ds, \quad \text{where } \mathcal{K} \in L^2([0,1]\times[0,1]).$$

HINT: the composition of A and the Volterra operator is a Hilbert–Schmidt operator.

7.10.112. Let H be a separable Hilbert space. Prove that the set of operators having a left or right inverse is everywhere dense in $\mathcal{L}(H)$.

HINT: use the polar decomposition, observe that a nonnegative selfadjoint operator can be approximated by positive operators and consider partial isometries.

7.10.113. Let K be a compact operator on a complex Banach space X. Let us consider the operator $A = \lambda I + K$, where $\lambda \in \mathbb{C}$. Prove that for every $\varepsilon > 0$ there exists an invertible operator A_ε with $\|A - A_\varepsilon\| \leqslant \varepsilon$.

HINT: use that the spectrum of K is at most countable.

7.10.114. Give an example of a noncompact nonnegative selfadjoint operator A on a separable Hilbert space such that there exists an orthonormal basis $\{e_n\}$ for which $\lim\limits_{n\to\infty} \|Ae_n\| = 0$.

HINT: consider the operator on l^2 defined by the infinite matrix of the following form: its nonzero elements are diagonal blocks, the nth of which is of size $n\times$ with all entries $1/n$; see also [**251**, p. 273].

7.10.115. (i) Let $e_1, \ldots, e_n$ be the standard basis in $\mathbb{C}^n$ or $\mathbb{R}^n$. Then the function $\|A\|_{\{e_i\}} = \sum_{i=1}^n \|Ae_i\|$ is a norm on the space of operators. Show that for the nuclear norm (the sum of eigenvalues of $\sqrt{A^*A}$) one has $\|A\|_{\{e_i\}} \leqslant \sqrt{n}\|A\|_1$ and there is a symmetric nonzero operator A for which there holds the equality.

(ii) Give an example of a nuclear operator $A \geqslant 0$ on a complex separable Hilbert space such that there exists an orthonormal basis $\{e_n\}$ for which $\sum_{n=1}^\infty \|Ae_n\| = +\infty$.

HINT: (i) take any orthonomal basis $\{\varphi_i\}$ such that φ_1 has equal coordinates $n^{-1/2}$ and the operator A defined by $A\varphi_1 = \varphi_1$, $A\varphi_j = 0$ if $j > 1$; then $\|A\|_1 = 1$ and $\|Ae_i\| = n^{-1/2}$ for all i, hence $\|A\|_{\{e_i\}} = n^{1/2}$. On the other hand, if α_i are eigenvalues of $\sqrt{A^*A}$, then $\|A\|_{\{e_i\}} \leqslant n^{1/2}\Bigl(\sum_{i=1}^n \alpha_i^2\Bigr)^{1/2}$, which is estimated by $n^{1/2}\sum_{i=1}^n \alpha_i$. An example in (ii) is easily constructed by using (i).

7.10.116. Give an example of a bounded operator A on the *real* space l^2 that is not nuclear, although there exists an orthonormal basis $\{e_n\}$ such that $(Ae_n, e_n) = 0$ and hence $\sum_{n=1}^\infty |(Ae_n, e_n)| < \infty$.

HINT: write l^2 as a countable sum of two-dimensional planes in which A acts as rotations by $\pi/2$.

7.10.117. (i) Let E be a closed linear subspace in $L^2[0,1]$ such that $E \subset L^\infty[0,1]$. Prove that $\dim E < \infty$.

HINT: apply Corollary 7.10.28 to obtain the compactness of the projection onto E.

7.10.118. Let H be an infinite-dimensional complex separable Hilbert space and let $\mathcal{U}(H)$ be the set of all unitary operators on H. (i) Investigate whether the space $\mathcal{U}(H)$ with the operator norm is connected (a space is connected if it cannot be decomposed into nonempty disjoint open parts). (ii) Find the closure of $\mathcal{U}(H)$ in the weak operator topology and in the strong operator topology.

HINT: (i) represent a given unitary operator U as $\exp(iA)$ with a bounded selfadjoint operator A and consider the family $\exp(itA)$. (ii) Observe that for every operator A with $\|A\| \leqslant 1$ and any two finite orthonormal collections $e_1, \ldots, e_n$ and $\varphi_1, \ldots, \varphi_n$, one can find a unitary operator U such that $(Ue_i, \varphi_j) = (Ae_i, \varphi_j)$, $i, j = 1, \ldots, n$. Observe also that if $\|Tx\| = \|x\|$ for all x, then on every finite-dimensional subspace T coincides with a unitary operator.

7.10.119. Let X and Y be Banach spaces and let p be a cross-norm on the tensor product $X\otimes Y$, i.e., $p(x\otimes y) = \|x\|\,\|y\|$. Prove the inequality $\|u\|_\infty \leqslant p(u) \leqslant \|u\|_{\mathcal{N}}$.

7.10.120. Show that a Fredholm operator $T \in \mathcal{L}(X,Y)$ between Banach spaces X and Y has zero index precisely when $T = S + K$, where S is invertible and K is compact, moreover, in this case K can be chosen finite-dimensional.

7.10.121. Suppose that X and Y are Banach spaces, $T_n \in \mathcal{L}(X,Y)$, $S_n \in \mathcal{L}(Y,X)$, the operators T_n converge pointwise to T, and the operators S_n converge pointwise to S. Suppose that $S_nT_n = I + K_n$, $T_nS_n = I + H_n$, where the operators K_n and H_n are uniformly compact in the sense that if U_X is the unit ball of X and U_Y is the unit ball of Y, then the sets $K_n(U_X)$ are contained in a common compact set and similarly for the sets $H_n(U_Y)$. Prove that $\operatorname{Ind} T = \lim\limits_{n\to\infty} \operatorname{Ind} T_n$.

HINT: see [**281**, Theorem 19.1.10].

7.10.122. Let T and S be nuclear operators on a separable Hilbert space H and let $A, B \in \mathcal{L}(H)$ be such that $AB = I - T$, $BA = I - S$. Prove that $\operatorname{Ind} A = \operatorname{tr} S - \operatorname{tr} T$.
HINT: see [**281**, Proposition 19.1.14].

7.10.123. Suppose that A is a nuclear operator on a separable Hilbert space H and operators $P_n, Q_n \in \mathcal{L}(H)$ converge pointwise to I. Prove that $\operatorname{tr}(P_n A Q_n^*) \to \operatorname{tr} A$.

7.10.124.* Let V be the Volterra operator of indefinite integration on $L^2[0,1]$. Let H_a be the subspace of functions vanishing on $[0,a]$, $a \in [0,1]$. Prove that V has no closed invariant subspaces different from H_a.
HINT: see [**378**, p. 280].

7.10.125. (Gleason's theorem) Let H be a separable complex Hilbert space, $\mathcal{P}$ the set of all orthogonal projections in H and $\mu\colon \mathcal{P} \to [0,1]$ a function such that $\mu(P) \leqslant \mu(Q)$ whenever $P \leqslant Q$ and $\mu\bigl(\sum_{i=1}^\infty P_i\bigr) = \sum_{i=1}^\infty \mu(P_i)$ if orthogonal projections P_i are pairwise orthogonal. Prove that $\mu(P) = \operatorname{tr} SP$, where S is a nonnegative trace class operator.

7.10.126. (Wigner's theorem) Let H be a separable complex Hilbert space, $\mathcal{P}$ the set of all orthogonal projections in H and $\xi\colon \mathcal{P} \to \mathcal{P}$ a mapping such that $\xi(I) = I$, $\xi(P) \leqslant \xi(Q)$ whenever $P \leqslant Q$ and for every sequence of pairwise orthogonal projections P_i the projections $\xi(P_i)$ are pairwise orthogonal and $\xi\bigl(\sum_{i=1}^\infty P_i\bigr) = \sum_{i=1}^\infty \xi(P_i)$. Prove that $\xi(P) = UPU^{-1}$, where U is a real-linear isometry that is either complex-linear or conjugate-linear.

7.10.127. Let X be a complex Banach space, $A \in \mathcal{L}(X)$ and let λ_0 be a pole of order m of the operator function $\lambda \mapsto R_\lambda(A) = (A - \lambda I)^{-1}$. Prove that λ_0 is an eigenvalue and for all $n \geqslant m$ one has the decomposition $X = \operatorname{Ker}(A - \lambda_0 I)^n \oplus (A - \lambda_0 I)^n(X)$.
HINT: see [**640**, Chapter VIII, §8].

7.10.128. Let H be a separable Hilbert space that is a dense linear subspace in a Banach space E such that the identity mapping $H \to E$ is continuous. Then the continuous embedding $E^* \to H^* = H$ defines the so-called triple of spaces $E^* \subset H \subset E$.

(i) Let A be a compact selfadjoint operator on H such that $A(H) \subset E^*$ and A is compact as an operator with values in E^*. Prove that the expansion $Ax = \sum_{n=1}^\infty \alpha_n (x, e_n) e_n$ in the eigenbasis $\{e_n\}$ of A converges with respect to the operator norm on $\mathcal{L}(H, E^*)$.

(ii) Suppose in addition that A extends to a compact operator from the space E to E^*. Prove that the indicated expansion also converges with respect to the operator norm on $\mathcal{L}(E, E^*)$.

(iii) Let (Ω, μ) be a probability space and let $\mathcal{K}\colon \Omega \times \Omega \to \mathbb{R}^1$ be a bounded measurable function such that $\mathcal{K}(t,s) = \mathcal{K}(s,t)$. Let $\{\lambda_i\}$ and $\{e_i\}$ be the eigenvalues and eigenfunctions of the operator T on $L^2(\mu)$ given by the integral kernel $\mathcal{K}$. Suppose that T is compact also as an operator from $L^1(\mu)$ to $L^\infty(\mu)$. Prove that

$$\lim_{n\to\infty} \operatorname{esssup}_{t\in\Omega} \operatorname{esssup}_{s\in\Omega} \Bigl|\mathcal{K}(t,s) - \textstyle\sum_{i=1}^n \lambda_i e_i(t) e_i(s)\Bigr| = 0.$$

HINT: see [**349**, Chapter III, §9].

7.10.129. Let A be a selfadjoint operator on a separable Hilbert space H with the resolution of the identity Π_0, $T \in \mathcal{L}(H)$.

(i) Prove that $AT = TA$ precisely when $T\Pi_0(\lambda) = \Pi_0(\lambda)T$ for all $\lambda \in \mathbb{R}^1$.

(ii) Prove that T commutes with all bounded operators commuting with A precisely when $T = f(A)$, where f is a bounded Borel function.

HINT: (i) use that in the strong operator topology A is a limit of linear combinations of operators $\Pi_0(\lambda)$ and that $\Pi_0(\lambda)$ is a limit of polynomials of A. (ii) Use Lemma 7.10.5 and observe that A is a function of a selfadjoint operator possessing a cyclic vector.

7.10.130.* Prove that the identity embedding $l^1 \to l^2$ is absolutely summing.

7.10.131. Let S be a countable union of compact sets in l^2. Prove that there is a unitary operator U (orthogonal in the real case) such that $U(S) \cap S$ is either empty or the zero element.

HINT: observe that for compact sets K_n there are numbers $\varepsilon_n > 0$ such that the union of $\varepsilon_n K_n$ has compact closure, hence is contained in a compact ellipsoid and consequently in the range of a compact operator, then apply Theorem 7.10.18.

7.10.132. (Kalisch [**691**]) Let us consider the operator

$$Kx(t) = tx(t) - \int_0^t x(s)\,ds$$

on the real space $L^2[0,1]$. Prove that for every nonempty compact set S in $\mathbb{R}^1$ there exists a closed subspace $H \subset L^2[0,1]$ such that $K(H) \subset H$ and the spectrum of K on H coincides with S and consists of eigenvalues.

7.10.133. Let E be a Banach lattice, $A \in \mathcal{L}(E)$ and $A \geqslant 0$ (see §6.10(v)). (i) Show that the spectral radius $r(A)$ is contained in $\sigma(A)$. (ii) Prove the Krein–Rutman theorem: if the operator A is compact and its spectral radius $r(A)$ is positive, then $r(A)$ is an eigenvalue and there exists an eigenvector $v \geqslant 0$.

HINT: see [**141**, Theorem 19.2], [**533**, p. 265], [**534**, Chapter V].

7.10.134.* Let H be a complex Hilbert space and let $A \in \mathcal{L}(H)$. Prove that the set $\{(Ax,x)\colon \|x\| = 1\}$ is convex in $\mathbb{C}$.

HINT: see [**251**].

7.10.135. (F. A. Sukochev) Let A and B be two nonnegative nuclear operators on a Hilbert space H. Then the operator $\sqrt{A^2+B^2}$ is nuclear. More generally, if $A_1,\ldots,A_n$ are nonnegative nuclear operators, then the operator $\sqrt{A_1^2+\cdots+A_n^2}$ is nuclear.

HINT: It suffices to prove the second assertion and apply induction on n. It is readily seen that for any 2×2-matrix M the operators $M\otimes A$ and $M\otimes B$ are nuclear on $\mathbb{C}^2\otimes H$. Let e_{ij} denote the 2×2-matrix with zero entries except for 1 at the intersection of the array number i and the column number j. Then the operator $T = e_{11}\otimes A + e_{21}\otimes B$ is nuclear. Since $T^* = e_{11}\otimes A + e_{12}\otimes B$, we have $T^*T = e_{11}\otimes(A^2+B^2)$. Hence $|T| = e_{11}\otimes\sqrt{A^2+B^2}$ is nuclear. It follows that $\sqrt{A^2+B^2}$ is nuclear as well.

7.10.136. (Brown, Pearcy [**678**]) Let A and B be bounded operators on a Hilbert space H. Then the spectrum of $A\otimes B$ on the Hilbert tensor product is $\sigma(A)\sigma(B)$.

7.10.137. (Brown, Pearcy [**677**]) Let H be a complex infinite-dimensional separable Hilbert space. An operator $A \in \mathcal{L}(H)$ is a commutator, i.e., has the form $A = ST - TS$ with $S,T \in \mathcal{L}(H)$, if and only if it is not of the form $\lambda I + K$, where $\lambda \neq 0$ and K is a compact operator.

7.10.138. (Brown, Pearcy, Salinas [**680**]) If T is a noncompact bounded linear operator on a separable Hilbert space H, then there exists a bounded nilpotent operator N on H (even with $N^3 = 0$) such that $N + T$ is invertible.

7.10.139. (Brown, Pearcy [**679**]) Every bounded operator T on a Hilbert space H with $\dim H > 1$ has the form $T = PAQ - QAP$, where P, A and Q are invertible operators.

7.10.140. Let A and B be positive operators on a Hilbert space H. Show that there is a positive operator T such that $B = TAT$.

HINT: consider $T = A^{-1/2}\bigl(A^{1/2}BA^{1/2}\bigr)^{1/2}A^{-1/2}$.